# Revival of a Classical Topic in Differential Geometry: Envelopes of Parameterized Families of Curves and Surfaces

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Classical topics in Differential Geometry like the study of 1-parameter families of curves and surfaces in 3D space and their envelopes have been abandoned in the past for various reasons [4]. Nevertheless, this topic has a great interest in mathematics and in applied science. The topic has lots of applications in science and engineering - caustics and wave fronts, i.e. Geometrical Optics and Theory of Singularities, robotics and kinematics, rigid motion in 2-space and in 3-space, collision avoidance, etc. Envelopes can be studied with paper-and-pencil together with both a Computer Algebra System (CAS) and a Dynamical Geometry System (DGS); see recent papers in [5]. For this work, we used more than one package.

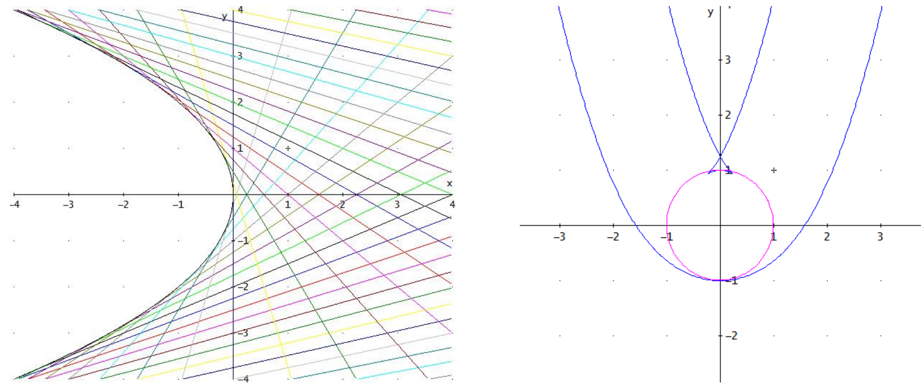
The authors performed similar work for another topic in classical Differential Geometry, namely isoptic curves of a given plane curve; see [1]. The influence of the technology was important. Here,as we will see, dynamical features are central.

Let be given a family of plane curves by an equation of the form $f(x,y,c) = 0$, where $c$ is a real parameter. An envelope of the family, if it exists, is a curve tangent to every curve in the family. It can be shown that this envelope is the solution set of the system of equations

$$\begin{cases} f(x,y,c) & = 0 \\ \frac{\partial f}{\partial c} f(x,y,c) & = 0 \end{cases} \tag{1}$$

Figure 1 shows the envelope of the family of lines given by the equation $x + cy = c^2$, where $c$ is a real parameter (it is the parabola whose equation is $y = -x^2/4$. and the envelope of the family of circles with radius 1 and center on the parabola whose equation is $y = x^2$ (here the result has two components, each component is an envelope of the family of circles).

In both cases, the usage of a slider bar enables to build envelopes experimentally. The experimental study may enhance the understanding either of the possible non-existence or of the possible non-uniqueness of an envelope. An example is displayed in Figure 2 for the family of circles whose radius is equal to 1 and whose center runs on an ellipse. On the left a partial construction is shown, a global construction appears on the right. This drawing has been obtained using the slider bar,



(a) Lines (b) Circles

Figure 1: Exploration of an envelope in the plane

which yields a uniform spacing between circles. Another possibility offered by the software is to move the center of the circle along the ellipse. In this case, spacing between neighboring circles is not uniform, the appearance of the envelopes being thus slightly different.

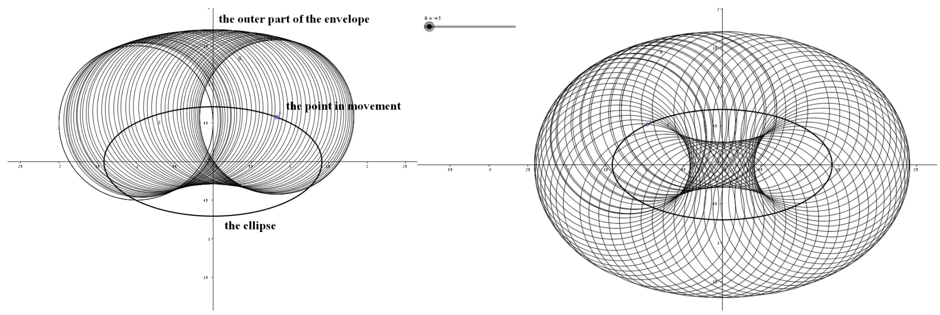



Figure 2: Dynamical exploration of the envelope of a family of circles

The transition to parameterized families of curves and surfaces in 3D space rely on the same techniques. For a 1-parameter family of surfaces, the defining equations for an envelope are now:

$$\begin{cases} f(x,y,z,c) & = 0 \\ \frac{\partial f}{\partial c} f(x,y,z,c) & = 0 \end{cases} \tag{2}$$

New issues have to be dealt with: the general visualization problems in this case, the availability of appropriate features in the software, etc. For the family of surfaces given by the equation $x + cy + c^2 z = c^3$, where c is a real parameter, an envelope can be found, shown in Figure 3. A cuspidal edge appears, as for every 1-parameter family of planes. In order to understand the surface visually, dynamical features of the software are a must. Otherwise, at least two stills pictures have to been plotted. In particular in such a case, the structure of the envelope as a ruled surface can be studied using technology. This topic is sometimes not easy for beginning students. We could check the influence of the choice of the mesh for plotting surfaces in 3D space on the students' understanding. For example, in Figure 3, lines can be seen who are tangent to the cuspidal edge; this is a central feature of such an envelope.

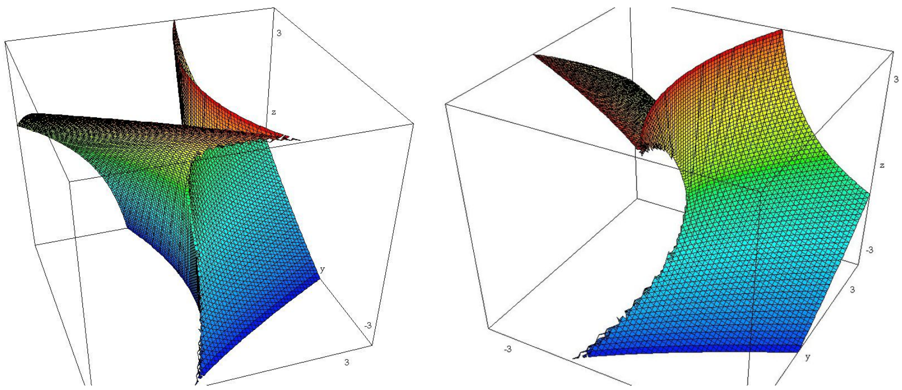


Figure 3: Exploration of the envelope of a family of planes

Such a study provides an opportunity to discover new topics beyond the scope of the regular curriculum, sometimes together with applications to practical situations. New computation skills with technology may be developed, in particular for the experimental aspect of the work (e.g., exploring the existence of cusps, as in Figure 1b). For this, the availability in the software of a slider bar is a central issue. Moreover, ability to switch between different registers of representation may be improved, within mathematics themselves (parametric vs implicit) and with the computer (algebraic, graphical, etc.).

An envelope may not exist (e.g. for a family of lines where the coefficients of the equations are affine functions of the parameter). These issues have been observed by the authors in sessions for in-service teachers at the Weizmann Institute of Science.

The algebraic engine we used in different CAS was the commands based on computations of Gröbner bases in order 1) to solve the given system of equations, which yields a parametric representation of the envelopes, and 2) to look for an implicitization of this parametric representation. The Gröbner bases algorithms have been widely used by Pech [2] for other geometric problems. When such an implicitization is not to be found, algorithms exist for an approximate implicitization (see [3]).

## References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