Contemporary interpretation of a historical locus problem with an unexpected discovery

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This talk is aimed primarily as a report on experience in the use of computer algebra (CAS) and dynamic geometry (DGS) systems in the teaching of future mathematics teachers. Specifically it deals with the detailed analysis of a historical locus problem selected from the Latin book *Exercitationes Geometricae* by Ioannis Holfeld, published by the Jesuit College of St. Clement in Prague in 1773 [3]; namely problem number 35 from a total of 47 solved problems that are presented in this book.

First, the original solution to problem 35 will be introduced. Then, it will be resolved using current methods supported by the use of GeoGebra and wxMaxima software. The different approach to the locus problem compared to the original solution allows us to reveal a more complete solution to the problem, which includes the surprising curve of a pretzel shape (a similar but not identical curve was mentioned in [1], see also `http://mathworld.wolfram.com/KnotCurve.html`). Finally we will derive both the algebraic equation and the parametric equations of the curve. All with the support of the mentioned software, as we solve it with students of mathematics teaching in the course of algebra and geometry.

Assignments of the problems in the book, as well as their solutions, illustrate the method of solving geometric problems typical for mathematics of the 17th and 18th centuries. Most of the problems, despite their age, are still attractive and are worth resolving with the help of contemporary methods. More information about the exercises, methods of their solutions and about the author of the book can be found in [2] and [4]. A copy of the original Latin assignments of the presented problem can be found at `http://www.pf.jcu.cz/~hasek/Holfeld`.

Problem 35: *Given a circle with a diameter MP (see Fig. 1); construct a radius AB to this circle and a line segment BO perpendicular to MP so that* $MO : AO = r : BC$ *(r is the radius of the circle). Find the locus of point C. (Remark: Length of the segment BC is the fourth proportional of lengths MO, AO and r.)* ([3], p. 41, *Problema 35*).

I. H. begins his solution by labeling lengths of selected segments; $AD = x$, $DC = y$, $AB = r$, $OM = z$. Then, using the similarity of triangles accompanied by the right triangle altitude theorem (also known as 'geometric mean theorem'), however, without mentioning the use of it, he derives the locus equation $y^2 = r^2 - 2rx$ (considering the configuration of the coordinate axes in Fig. 2), that corresponds

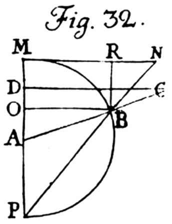

Figure 1: Illustration of the assignment of problem 35

to the parabola. Its plot for the particular value of r is shown in Fig. 2, left. On the right in the same figure the result is presented of the use of GeoGebra's tool "Locus" to find the desired locus curve. Considering all possible positions of point C on the ray AB with respect to point B we receive a surprising result. The curve consists of two parts; parabola, which was identified as the solution by I. H., and a curve that looks like a pretzel. To learn more about these curves we represent

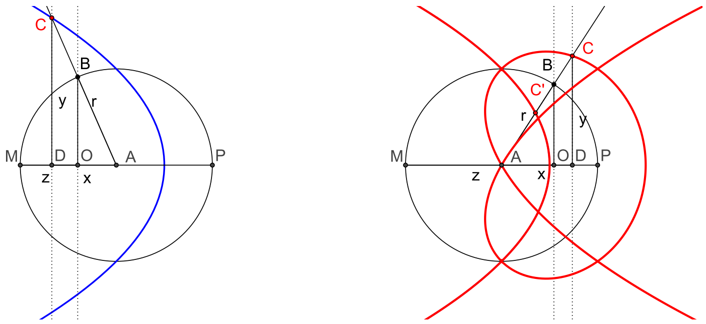

Figure 2: Solution of problem 35 according to I. H. (left) and the solution by means of GeoGebra's tool "Locus" (right)

the task by means of the system of nonlinear equations. Its solution by elimination leads to the sixth degree algebraic equation in the variables x and y, the polynomial of which can always be factored into the product of two polynomials of the second and fourth degree respectively (1).

$$(y^2+2rx-r^2)(y^4+x^2y^2-2rxy^2-2rx^3+3r^2x^2-r^2y^2)=0, \qquad (1)$$

The second degree factor corresponds to the parabola (the solution given by I. H.) and the fourth degree factor defines the 'pretzel' curve, both curves shown in Fig. 2, right.

This unique event of the discovery of a new curve motivates us to explore its properties. Above all, we will focus on its parameterization. With the use of the computer we can introduce students to the principles of the parameterization of curves (see Fig. 3), illustrate it in examples and let them find the parametric

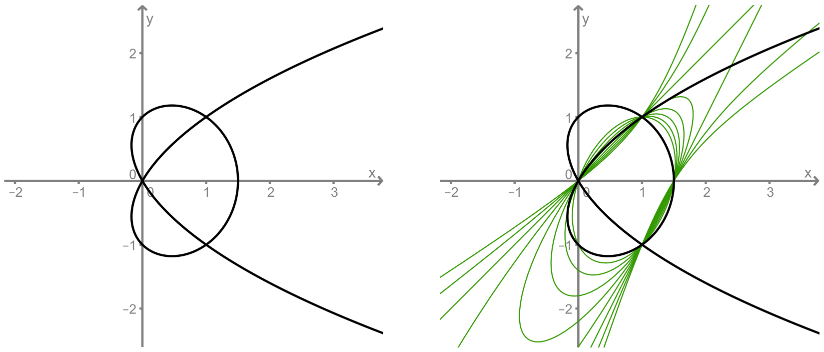

Figure 3: The discoveried curve and its parametrization by a system of conics

equations of the new curve that are for example as follows

$$x = \frac{t^4 - 4t^2 + 3}{2t^2 + 2}, \quad y = \frac{t^3 - 3t}{t^2 + 1}. \tag{2}$$

We have experienced that the return to historical tasks can be inspiring and beneficial if the solver is equipped with analytical methods and the proper algebraic and dynamic geometry software. Through the study of the presented historical geometry problem, the solution of which has not been described in any textbook since the 18th century, students develop or practice their knowledge of the geometry of curves, polynomial algebra and the effective use of mathematical software.

References

[1] H.M. Cundy, *Mathematical models*, 2nd ed., Oxford University Press, Oxford (1961).

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[3] I. Holfeld, *Exercitationes Geometricae*, Charactere Collegii Clementini Societas Jesu, Praha (1773).

[4] J. Zahradník, *Problémy z geometrie ve sbírce Ioannise Holfelda Exercitationes geometricae* (in Czech), Sborník 34. mezinárodní konference Historie matematiky, Poděbrady, 23. - 27. srpna 2013, Matfyzpress, Praha, p. 191 (2013).