A Constructive Proof of Feuerbach's Theorem Using a Computer Algebra System

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Feuerbach's Theorem states that the midpoints of the three sides, the base points of the three heights, and the midpoints of the line segments between the corners of a triangle and the intersection of the heights are on a circle. This talk offers a constructive proof. It is known that algebraic expression

$$x^2 + y^2 + dx + ey + f = 0 \tag{1}$$

represents a circle centered at $(-\frac{d}{2}, -\frac{e}{2})$ with radius

$$r^2 = \frac{d^2 + e^2 - 4f}{4} \tag{2}$$

provide (2) is positive. Three points among nine stated in the theorem are chosen to form a system of linear equations from (1). The values of d, e and f are determined by solving the equations. With the solution, (2) is shown to be positive which implies (1) indeed represents a circle. We then proceed to verify that the coordinates of the remain six points statisfy (1). Hence all nine points are on the same circle.

References

[1] B. Spain, *Analytical Conics* pp. 21-23 (2007).

[2] J. Gullberg, *Mathematics From the Birth of Numbers* p. 433 (1997).