Plotting technologies for the study of functions of two real variables

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Students learning towards a degree in a STEM related domain learn quite early a course in Advanced Calculus, i.e. a course where the main object of study are multivariable functions. It happens that students cannot *see* how these objects behave. In particular, difficulties appear in classroom for functions having different limits at a point, according to the path approaching the point. Therefore dynamical visualization techniques are important.

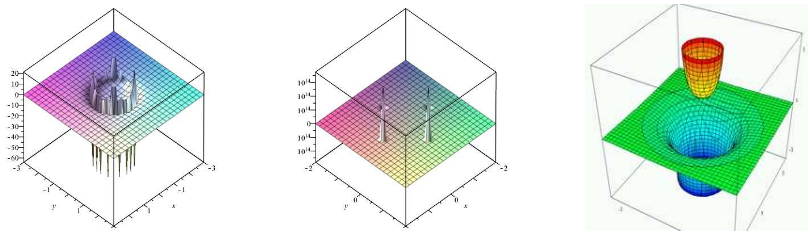
The study of functions of two real variables can be supported by visualization using a Computer Algebra System (CAS). Contour plots were the first type of graphic representations. With the development of scientific computing, 3D plots were introduced and plotting the graph of a two-variable function has been made possible, including parametric plotting and implicit plotting.

Depending both on the hardware and on the software, constraints exist, making sometimes the plots non accurate. For example, the choice of the mesh (using a triangulation of the domain, or using geodesics on the surface, etc.) has a great influence on the representation. Interpolations are performed, which may hide discontinuities. Other constraints come from the need to control the geometric transformations of the plotted surface: transformations such as zooming, rotation around a given axis, displacement along a given path are examples where the visualization device needs to understand the mathematical behavior of the function.

Here are examples of non-accurate plots, obtained with a brute force usage of commands, without a suitable analysis of the function. The function we tried to plot in Figure 1 is given by $f(x,y) = \frac{1}{1-(x^2+y^2)}$. The only difference in the command that has been entered is the x-intervals and the y-intervals, $[-2,2]$ in (a) and $[-3,3]$ in (b). The rightmost plot is quite accurate, it uses other techniques.

In the unaccurate plots, interpolations hide the actual discontinuities in different ways. Regular zooming cannot solve this problem, as it inflates the cells but does not recompute the needed numerical data.

Rotation around a given axis often masks discontinuities of the function and can also provide strange plots, not compatible with the mathematics. In recent years, point based geometry has gained increasing attention as an alternative surface representation, both for efficient rendering and for flexible geometry process-



(a) (b) (c)

Figure 1: Strange plots

ing of complex surfaces. More sophisticated representations that use lighting effects and virtual reality are available. We analyze the efficiency of the different representations with respect to the mathematical behavior of a function of two real variables.

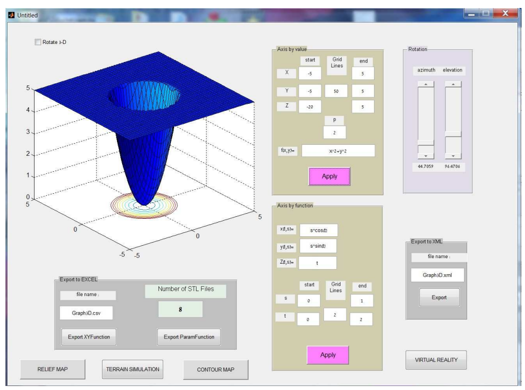


Figure 2: Interface

An interface written in Matlab 14 has been developed for producing different representations of a given explicit function of two variables; (see Figure 2). After having received an analytic expression for the function, the software produces four types of representation, namely:

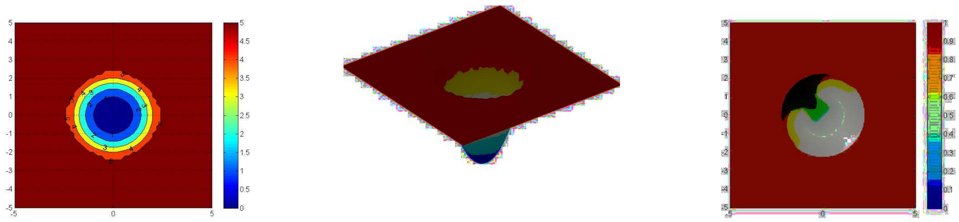
1. A color contour map of the function.

2. A three dimensional plot of function.

3. A relief terrain map corresponding to the height of the function.

4. A virtual reality scene where the function is a terrain and the user is flying over it.

The first three representations are illustrated for the function given by $f(x,y) = x^2 + y^2$ in Figure 3. Note that this function has a discontinuity at $(0,0)$.



(a) Contours (b) 3D plot (c) Terrain relief

Figure 3: Color plottings

The virtual reality scene is shown for the same function in Figures (4 and 5).

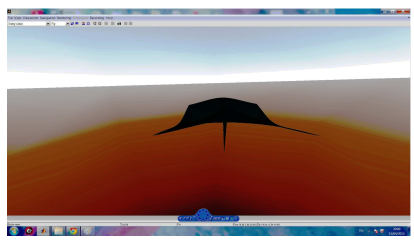


Figure 4: Virtual scene

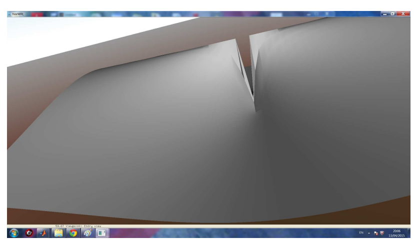


Figure 5: Path along a discontinuity

Comparison between the different kinds of representation is done with respect to the accuracy and the teaching of:

1. The global plotting of the function.
2. The appearance of the existing discontinuities and the non-appearance of non-existing discontinuities.

3. The visualization of directional derivatives.
4. The different types of optima, such as maximum. minimum points and saddle points.

Some of the issues that are to be discussed are as follows, all of them have an influence on the joint work of the students and the educator :

- The domain of the given function may be unbounded. Nevertheless, the plotting domain is always bounded; this is one of the constraints mentioned previously.
- Meshing techniques versus isoclines plotting.

This comparison and its outcome were an incitement to the usage of virtual reality in order to represent a function of two variables. In particular, the VR device under development is designed to generate paths on the surface under study and to replace classical zooming which only inflates the existing cells of the mesh by a re-computation of the data required for the plotting. This may allow to visualize clearly discontinuities and different types of extrema.

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