Constructing Rational Gram–Schmidt Problems and QR Problems

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A standard topic in Linear Algebra is the Gram-Schmidt process. It is equivalent to obtaining the QR factoring of a matrix. When books use the name Gram-Schmidt, they start with a set of vectors; when they use the term QR decomposition, they start with a matrix. They are equivalent because each matrix column is a vector. The aim in either case is to make each column of unit length, in the 2-norm, and also make each column orthogonal to all others. The resulting matrix is called orthonormal (or orthogonal). So a matrix A is factored (decomposed) as

$$A = QR \, ,$$

where Q is orthonormal and R is upper triangular, and Q has the property that $Q^T Q = I$, that is, its inverse equals its transpose. In *numerical* linear algebra, an important property is that Q can be regarded a rotation matrix, and therefore it does not amplify rounding errors the way that LU factoring does. The 2-norm is the square-root of the sum of squares of the components of a vector, and hence the Gram-Schmidt process includes many square-roots, which make exam questions painful, because the students get lost. How nice if all square roots were exact!

We all know $3^2 + 4^2 = 5^2$ (don't we); so the 2-norm of vector $[3,4]$ is 5. Some may know that $3^2 + 4^2 + 12^2 = 13^2$; so the 2-norm of $[3,4,12]$ is 13. We can use this to construct matrices with rational QR factors.

In general, we define a *pythagorean n-tuple* by the equation

$$x_1^2 + x_2^2 + \ldots + x_{n-1}^2 = x_n^2 \, ,$$

where the x_k are all integers. Then the above examples are a pythagorean triple and quadruple. A number of algorithms have been published to generate n-tuples [1], but we need *orthogonal* n-tuples. At the moment, the only way we have found to get the orthogonality property is brute force (exactly what computer algebra is good at). However, it works quite well, and we have successfully constructed 5-by-5 matrices with rational QR factors.

References

[1] Sophie Frisch & Leonid Vaserstein, *Polynomial parametrization of Pythagorean quadruples, quintuples and sextuples*, J. Pure Applied Algebra **216**, pp. 184–191 (2011).