

THE $S_{\mathbf{w}}^3$ SASAKI JOIN CONSTRUCTION

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ABSTRACT. The main purpose of this work is to generalize the $S_{\mathbf{w}}^3$ Sasaki join construction $M \star_1 S_{\mathbf{w}}^3$ described in the authors' 2016 paper when the Sasakian structure on M is regular, to the general case where the Sasakian structure is only quasi-regular. This gives one of the main results, Theorem 3.2, which describes an inductive procedure for constructing Sasakian metrics of constant scalar curvature. In the Gorenstein case ($c_1(\mathcal{D}) = 0$) we construct a polynomial whose coefficients are linear in the components of \mathbf{w} and whose unique root in the interval $(1, \infty)$ completely determines the Sasaki-Einstein metric. In the more general case we apply our results to prove that there exists infinitely many smooth 7-manifolds each of which admit infinitely many inequivalent contact structures of Sasaki type admitting constant scalar curvature Sasaki metrics (see Corollary 6.15). We also discuss the relationship with a recent paper of Apostolov and Calderbank as well as the relation with K-stability.

INTRODUCTION

It is often said that Sasaki geometry is the odd dimensional version of Kähler geometry; nevertheless, there are substantial differences. The most obvious difference concerns products. The product of Kähler manifolds (orbifolds) is Kähler. This simple formulation does not occur in Sasaki geometry for dimensional reasons. The closest analogue is the so-called Sasaki join operation which amounts to a product of the corresponding transverse Kähler structures and which by now has been developed in some detail [BG00b, BGO07, BG08, HS15, BTF16, BHLTF18]. However, the Sasaki join is much richer as we shall now discuss. The Kähler and Sasaki cones are analogous in the sense that they provide a family of Kähler (Sasaki) structures that are associated to a fixed underlying complex, (respectively, CR) structure. They are, however, very different in nature. The Kähler cone is a cone in the vector space of 2-dimensional de Rham cohomology classes; whereas, the Sasaki cone consists of nowhere vanishing Killing vector fields. Now consider the following examples. First, let (N, ω_N) be a Kähler-Einstein manifold or orbifold and consider the product with the standard complex projective line $(\mathbb{C}\mathbb{P}^1, \omega_{FS})$ with its Fubini-Study metric. Then there is a choice of positive integers l_1, l_2 such that $(N \times \mathbb{C}\mathbb{P}^1, l_1\omega_N + l_2\omega_{FS})$ is Kähler-Einstein. A similar analogue holds for the Sasaki join $M \star_1 S^3$. Now if we replace $(\mathbb{C}\mathbb{P}^1, \omega_{FS})$ by a weighted projective line $(\mathbb{C}\mathbb{P}^1[\mathbf{w}], \omega_{ext})$ with an extremal Kähler orbifold metric, there is no pair of integers l_1, l_2 nor deformation in the Kähler cone that produces a Kähler-Einstein orbifold metric on the product $N \times \mathbb{C}\mathbb{P}^1[\mathbf{w}]$. However, as shown explicitly in [BTF16] when M is a regular Sasaki-Einstein manifold, for each weight vector $\mathbf{w} = (w_1, w_2)$ there are positive integers l_1, l_2 such that the corresponding Sasaki join $M \star_1 S_{\mathbf{w}}^3$ admits a Sasaki-Einstein metric in the Sasaki cone. The purpose of this paper is to generalize the study of the

Date: September 27, 2021.

The authors were partially supported by grants from the Simons Foundation, CPB by (#519432), and CWT-F by (#422410).

$S_{\mathbf{w}}^3$ join construction in [BTF16] to the case where M in $M_{1,\mathbf{w}} = M \star_1 S_{\mathbf{w}}^3$ is an arbitrary quasi-regular Sasaki manifold or orbifold. This has important consequences for GIT stability in the Sasaki category that does not hold in the Kähler category. Namely, in a certain sense the $S_{\mathbf{w}}^3$ Sasaki join operation preserves K-stability for all weight vectors \mathbf{w} . This amounts to the join of an unstable, but relatively K-stable manifold, producing a K-stable Sasaki manifold.

The complexity of the multiplicative structure described above, the l -monoid, adds to the complexity of the moduli space of Sasakian structures which increases drastically with dimension. In dimension 3 things are well understood and there is a classification of Sasakian structures [Bel01, BG08]. In dimension 5 it is already much more complex. See Chapter 10 of [BG08] and Section 6 of [Boy19] for a recent update. There is, however, a classification of simply connected compact 5-manifolds due to Smale and Barden, not all of which admit Sasakian structures. Precisely which ones do and which don't is still an open question. Reducibility essentially begins in dimension 5 where the so-called cone decomposable Sasakian structures are lens space bundles over Riemann surfaces [BTF14a, BHLTF18]. See Corollary 3.5 below. In dimension 7 things become yet even more complex, since there are many irreducible Sasaki 5-manifolds. Given this situation it seems prudent to understand which Sasakian structures can be built up from lower dimensional ones and how stability propagates. This is precisely what the join accomplishes.

Acknowledgements. We thank Vestislav Apostolov and David Calderbank for many very helpful discussions and their interest in our work. We also thank the anonymous referees for their careful reading of our manuscript, and especially for catching an error in an earlier version of Proposition 2.20 and for clarifications in Section 5.

1. BRIEF REVIEW OF SASAKI GEOMETRY

Recall that a Sasakian structure on a contact manifold M^{2n+1} of dimension $2n + 1$ is a special type of contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with underlying almost CR structure (\mathcal{D}, J) where η is a contact form such that $\mathcal{D} = \ker \eta$, ξ is its Reeb vector field, $J = \Phi|_{\mathcal{D}}$, and $g = d\eta \circ (\mathbb{1} \times \Phi) + \eta \otimes \eta$ is a Riemannian metric. \mathcal{S} is a Sasakian structure if ξ is a Killing vector field and the almost CR structure is integrable, i.e. (\mathcal{D}, J) is a CR structure. We refer to [BG08] for the fundamentals of Sasaki geometry. We call (\mathcal{D}, J) a *CR structure of Sasaki type*, and \mathcal{D} a *contact structure of Sasaki type*. We shall always assume that the Sasaki manifold M^{2n+1} is compact and connected.

1.1. The Sasaki Cone. Within a fixed contact CR structure (\mathcal{D}, J) there is a conical family of Sasakian structures known as the Sasaki cone. We are also interested in a variation within this family. To describe the Sasaki cone we fix a Sasakian structure $\mathcal{S}_o = (\xi_o, \eta_o, \Phi_o, g_o)$ on M whose underlying CR structure is (\mathcal{D}, J) and let \mathfrak{t} denote the Lie algebra of a maximal torus \mathbb{T} in the automorphism group of \mathcal{S}_o . The (*unreduced*) *Sasaki cone* [BGS08] is defined by

$$(1) \quad \mathfrak{t}^+(\mathcal{D}, J) = \{\xi \in \mathfrak{t} \mid \eta_o(\xi) > 0 \text{ everywhere on } M\},$$

which is a cone of dimension $k \geq 1$ in \mathfrak{t} . The reduced Sasaki cone $\kappa(\mathcal{D}, J)$ is $\mathfrak{t}^+(\mathcal{D}, J)/\mathcal{W}$ where \mathcal{W} is the Weyl group of the maximal compact subgroup of $\mathcal{CR}(\mathcal{D}, J)$ which, as mentioned previously, is the moduli space of Sasakian structures with underlying CR

structure (\mathcal{D}, J) . However, it is more convenient to work with the unreduced Sasaki cone $\mathfrak{t}^+(\mathcal{D}, J)$. It is also clear from the definition that $\mathfrak{t}^+(\mathcal{D}, J)$ is a cone under the transverse scaling defined by

$$(2) \quad \mathcal{S} = (\xi, \eta, \Phi, g) \mapsto \mathcal{S}_a = (a^{-1}\xi, a\eta, g_a), \quad g_a = ag + (a^2 - a)\eta \otimes \eta, \quad a \in \mathbb{R}^+.$$

So $\mathfrak{t}^+(\mathcal{D}, J)$ is a cone and since the Reeb vector field ξ is Killing $\dim \mathfrak{t}^+(\mathcal{D}, J) \geq 1$. Moreover, it follows from contact geometry that $\dim \mathfrak{t}^+(\mathcal{D}, J) \leq n + 1$. When $\dim \mathfrak{t}^+(\mathcal{D}, J) = n + 1$ we have a toric contact manifold of Reeb type studied in [BM93, BG00a, Ler02, Ler04, Leg11a, Leg16]. In this case there is a strong connection between the geometry and topology of (M, \mathcal{S}) and the combinatorics of $\mathfrak{t}^+(\mathcal{D}, J)$. Much can also be said in the complexity 1 case ($\dim \mathfrak{t}^+(\mathcal{D}, J) = n$) [AH06].

We often have need to deform the contact structure $\mathcal{D} \mapsto \mathcal{D}_\varphi$ by a contact isotopy $\eta \mapsto \eta + d^c\varphi$ where $\varphi \in C^\infty(M)^\mathbb{T}$ is a smooth function invariant under the torus \mathbb{T} . We note that the Sasaki cone $\mathfrak{t}^+(\mathcal{D}, J)$ is invariant under such contact isotopies, that is, $\mathfrak{t}^+(\mathcal{D}_\varphi, J_\varphi) = \mathfrak{t}^+(\mathcal{D}, J)$. For each $\varphi \in C^\infty(M)^\mathbb{T}$, $\mathcal{D}_\varphi \rightarrow TM$ gives a splitting of the exact sequence

$$0 \rightarrow L_{\xi_\varphi} \rightarrow TM \rightarrow Q \rightarrow 0$$

with $J_\varphi = \Phi|_{\mathcal{D}_\varphi}$. Furthermore, each choice of Reeb vector field $\xi \in \mathfrak{t}^+(\mathcal{D}, J)$ gives rise to an infinite dimensional contractible space $\mathcal{S}(M, \xi)$ of Sasakian structures [BG08]. We shall often make such a choice $\mathcal{S} = (\xi, \eta, \Phi, g) \in \mathcal{S}(M, \xi)$ and identify it with the element $\xi \in \mathfrak{t}^+(\mathcal{D}, J)$.

2. THE S^3 -JOIN CONSTRUCTION

The join construction is the Sasaki analogue for products in Kähler geometry. It was first described in the context of Sasaki-Einstein manifolds [BG00b], but then developed more generally in [BGO07], see also Section 7.6.2. of [BG08]. Its relation to the de Rham decomposition Theorem and reducibility questions [HS15] were studied in [BHLTF18]. We consider the set \mathcal{SO} (respectively, \mathcal{SM}) of all Sasaki orbifolds (manifolds), respectively. \mathcal{SO} is the object set of a groupoid whose morphisms are orbifold diffeomorphisms. Moreover, \mathcal{SO} is graded by dimension and has an additional multiplicative structure described in Section 2.4 of [BHLTF18] which we call the *Sasaki l -monoid*. The positive irreducible generators in dimension 3 are of the form $\star_1 S_{\mathbf{w}}^3$ where the weight vector $\mathbf{w} = (w^0, w^\infty)$ has relatively prime components. We apply this to the submodule \mathcal{SM} which is not an ideal in general. However, as we shall see $\star_1 S_{\mathbf{w}}^3$ does give a map $\mathcal{SM} \rightarrow \mathcal{SM}$ when certain conditions hold. We have used the join construction [BTF14a, BTF15, BTF16] to produce new cscS metrics from known cscS metrics. In most cases (see [BTF19] for an exception) in constructing the join $M \star_1 S_{\mathbf{w}}^3$ we have assumed that the Sasakian structure on M is regular. This is a restrictive assumption which is not necessary. We only need the Sasakian structure \mathcal{S} to be quasiregular in which case it is described by an S^1 orbibundle $M \rightarrow N$ which has order $\Upsilon_{\mathcal{S}}$. The procedure as outlined for example in [BTF16] essentially works in the more general quasiregular case; however, one needs to proceed with a bit more care at several stages, in particular, in handling the smoothness conditions. Before discussing these conditions we briefly discuss the orbibundle $M \rightarrow N$. We assume that N is a normal (compact)

projective algebraic variety with a fixed orbifold structure which we write as the pair (N, Δ_N) where Δ_N is a sum of irreducible branch divisors, viz.

$$\Delta_N = \sum_{j=1}^k \left(1 - \frac{1}{m_j}\right) D_j$$

where $k = \dim \operatorname{Div}(N)$, the group of Weil divisors on N , m_j are the ramification indices, and $D_j \in \operatorname{Div}(N)$. The algebraic singular locus is also given an orbifold structure. They are finite cyclic quotient singularities.

By Theorem 4.3.15 of [BG08] the isomorphism classes of S^1 orbibundles over (N, Δ_N) are uniquely determined by their orbifold first Chern class denoted by $c_1^{orb}(M/N) \in H^2(N, \mathbb{Q})$. Furthermore, when M has a Sasakian structure, $c_1^{orb}(M/N)$ is an orbifold Kähler class with Kähler form denoted by ω_N .

Definition 2.1. We say that a compact Sasaki manifold $M = (M, \mathcal{S})$ with \mathcal{S} quasi-regular is *Sasaki primitive* or just *primitive* if there is no nontrivial finite cyclic Sasaki cover \tilde{M} of M . By a finite cyclic Sasaki cover we mean a Sasaki manifold $(\tilde{M}, \tilde{\mathcal{S}})$ such that $M = \tilde{M}/G$ with G a finite cyclic subgroup of the circle group induced by the Reeb vector field ξ . Here \mathcal{S} is the Sasakian structure on the quotient M that is naturally induced from $\tilde{\mathcal{S}}$.

Remark 2.2. It is well known that a compact regular Sasaki manifold (M, \mathcal{S}) is primitive if and only if the induced Kähler class $[\omega_N]$ is a primitive element of $H^2(N, \mathbb{Z})$. Moreover, there is a family $\{(M_n, \mathcal{S}_n)\}_{n \in \mathbb{Z}^+}$ of Sasaki manifolds with $(M_1, \mathcal{S}_1) = (M, \mathcal{S})$ associated to the primitive Kähler class $[\omega_N]$ whose members have Kähler class $n[\omega_N]$ and $M_n = M/\mathbb{Z}_n$ where \mathbb{Z}_n acts freely on M and there is a short exact sequence

$$0 \longrightarrow \pi_1(M) \longrightarrow \pi_1(M_n) \longrightarrow \mathbb{Z}_n \longrightarrow 1.$$

However, an analogous result does not hold for quasi-regular Sasaki manifolds unless $\gcd(n, \Upsilon_N) = 1$ where Υ_N is the order of the orbifold N . This is easily seen by the following example. Consider (N, Δ) with N smooth and choose a primitive Kähler class $[\omega_N] \in H_{orb}^2(N, \mathbb{Z})$. Then $\Upsilon_N[\omega_N]$ is a primitive Kähler class in $H^2(N, \mathbb{Z})$ and as such the total space of the corresponding S^1 bundle over N has a regular primitive Sasakian structure, but $\Upsilon_N[\omega_N]$ is clearly not primitive as an orbifold Kähler class.

2.1. The Construction. We now turn to the $S_{\mathbf{w}}^3$ join construction. Let M be a compact quasi-regular Sasaki manifold whose S^1 action generated by its Reeb vector field is denoted by $\mathcal{A}(\theta, x)$. The assumption for the rest of the paper is that $c_1^{orb}(M/N) = [\omega_N] \in H^2(N, \mathbb{Q})$ is a primitive orbifold Kähler class, where ω_N denotes the transverse Kähler form on N . Likewise, $c_1^{orb}(S_{\mathbf{w}}^3/\mathbb{C}\mathbb{P}^1[\mathbf{w}]) = [\omega_{\mathbf{w}}] = [\frac{\omega_{FS}}{w^0 w^\infty}]$, where ω_{FS} is the Kähler form of the standard Fubini-Study metric on $\mathbb{C}\mathbb{P}^1$ such that $[\omega_{FS}] \in H^2(\mathbb{C}\mathbb{P}^1, \mathbb{Z})$ is primitive and $\omega_{\mathbf{w}}$ is the transverse extremal Kähler form on $\mathbb{C}\mathbb{P}^1[\mathbf{w}]$ of the canonical extremal Sasaki structure on $S_{\mathbf{w}}^3$.

For any pair of relatively prime positive integers l^0, l^∞ we define the $S_{\mathbf{w}}^3$ join with M as the quotient $M_{l^0, l^\infty, \mathbf{w}}$ of

$$(3) \quad S^1 \longrightarrow M \times S^3 \longrightarrow M_{l^0, l^\infty, \mathbf{w}}$$

where the S^1 action is given by

$$(4) \quad (x; z_1, z_2) \mapsto (\mathcal{A}(l^\infty \theta, x); e^{-il^0 w^0 \theta} z_1, e^{-il^0 w^\infty \theta} z_2)$$

with $|z_1|^2 + |z_2|^2 = 1$. The join is then constructed from the following commutative diagram

$$(5) \quad \begin{array}{ccc} M \times S_{\mathbf{w}}^3 & & \\ & \searrow \pi_L & \\ & & M_{1,\mathbf{w}} \\ & \swarrow \pi_1 & \\ (N, \Delta_N) \times \mathbb{C}\mathbb{P}^1[\mathbf{w}] & & \end{array}$$

$\downarrow \pi_2$

where the π_2 is the product of the projections of the standard Sasakian projections $\pi_M : M \rightarrow (N, \Delta_N)$ and $S_{\mathbf{w}}^3 \rightarrow \mathbb{C}\mathbb{P}^1[\mathbf{w}]$. The circle action (4) on $M \times S^3$ is generated by the vector field

$$(6) \quad L_{1,\mathbf{w}} = \frac{1}{2l^0} \xi_M - \frac{1}{2l^\infty} \xi_{\mathbf{w}},$$

and its quotient orbifold $M_{1,\mathbf{w}}$, which is called the $\mathbf{1} = (l^0, l^\infty)$ -join of M and $S_{\mathbf{w}}^3$ and denoted by $M_{1,\mathbf{w}} = M \star_{\mathbf{1}} S_{\mathbf{w}}^3$, has a naturally induced quasi-regular Sasakian structure $\mathcal{S}_{1,\mathbf{w}}$ with contact 1-form $\eta_{1,\mathbf{w}}$ and Reeb vector field

$$(7) \quad \xi_{1,\mathbf{w}} = \frac{1}{2l^0} \xi_M + \frac{1}{2l^\infty} \xi_{\mathbf{w}}.$$

On the join $M_{1,\mathbf{w}} = M \star_{\mathbf{1}} S_{\mathbf{w}}^3$ there is a direct sum decomposition of the tangent bundle [HS15, BHLTF18]

$$(8) \quad TM_{1,\mathbf{w}} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus L_\xi$$

which for $i = 1, 2$ gives rise to two foliations $\mathcal{E}_i = \mathcal{D}_i \oplus L_\xi$ of $M_{1,\mathbf{w}}$ whose leaves are totally geodesic Sasaki submanifolds isomorphic, up to transverse scaling, as Sasakian structures to M and $S_{\mathbf{w}}^3$, respectively. Note that the intersection $\mathcal{E}_1 \cap \mathcal{E}_2$ is just \mathcal{F}_ξ , and both the transverse metric g^T and the contact bundle $\mathcal{D}_{1,\mathbf{w}} = \ker \eta_{1,\mathbf{w}}$ split as direct sums. The natural numbers $l^0, l^\infty, w^0, w^\infty$ are generally contact invariants. Since the CR-structure $(\mathcal{D}_{1,\mathbf{w}}, J_{\mathbf{w}})$ is the horizontal lift of the complex structure on $N \times \mathbb{C}\mathbb{P}^1[\mathbf{w}]$, this splits as well. The choice of \mathbf{w} determines the transverse complex structure J .

2.2. Some Elementary Topology. Generally, the join is an orbifold; however, from Proposition 7.6.6 in [BG08] we have

Lemma 2.3. *If (M, \mathcal{S}) is a quasi-regular Sasaki manifold of order $\Upsilon_{\mathcal{S}}$, then the join $M_{1,\mathbf{w}} = M \star_{\mathbf{1}} S_{\mathbf{w}}^3$ is smooth if and only if*

$$(9) \quad \gcd(l^\infty \Upsilon_{\mathcal{S}}, l^0 w^0 w^\infty) = 1$$

where the order $\Upsilon_{\mathcal{S}}$ is precisely the order Υ_N of the quotient orbifold N , i.e. $\Upsilon_{\mathcal{S}} = \Upsilon_N$.

From the long exact homotopy sequence of the fibration (3) and the well known Hurewicz Theorem we deduce

Proposition 2.4. *Let M be a simply connected quasi-regular Sasaki manifold with $\pi_2(M) = \mathbb{Z}^k$. Then*

- (1) $M_{1,\mathbf{w}}$ is simply connected;
- (2) $\pi_2(M_{1,\mathbf{w}}) = \mathbb{Z}^{k+1}$;
- (3) $H_2(M_{1,\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}^{k+1}$.

The join construction provides a family of contact structures of Sasaki type on a family of smooth manifolds whose cohomology ring can, in principle, be computed if one knows the cohomology ring of M .

2.3. The \mathbf{w} -Sasaki Cone $\mathfrak{t}_{\mathbf{w}}^+$. The Sasakian structures that are most accessible through this construction are the elements of the 2-dimensional subcone $\mathfrak{t}_{\mathbf{w}}^+$ of the Sasaki cone $\mathfrak{t}_{1,\mathbf{w}}^+$ known as the \mathbf{w} -subcone. If N has no Hamiltonian symmetries then $\mathfrak{t}^+ = \mathfrak{t}_{\mathbf{w}}^+$. On $M \times S^3$ we have the Lie algebra $\mathfrak{t}_M \oplus \mathfrak{t}_2$ which induces the Lie algebra isomorphism

$$(10) \quad \mathfrak{t}_{M,1,\mathbf{w}} \approx (\mathfrak{t}_M \oplus \mathfrak{t}_2) / \{L_{1,\mathbf{w}}\}$$

on $M_{1,\mathbf{w}}$ where $\{L_{1,\mathbf{w}}\}$ denotes the one dimensional ideal generated by the vector field $L_{1,\mathbf{w}}$. The Lie algebra \mathfrak{t}_M can be written as $\mathfrak{t}_M = \{\xi_M\} + \mathfrak{h}_M$ where \mathfrak{h}_M is the horizontal lift to M of a maximal commutative Lie algebra of Hamiltonian vector fields on N . Likewise $\mathfrak{t}_2 = \{\xi_{\mathbf{w}}\} + \mathfrak{h}_{\mathbf{w}}$ where $\mathfrak{h}_{\mathbf{w}}$ is the horizontal lift of the 1-dimensional Lie algebra generated by a Killing vector field on $\mathbb{C}\mathbb{P}^1[\mathbf{w}]$. Now the foliations \mathcal{E}_i on $M_{1,\mathbf{w}}$ give rise to Lie algebra monomorphisms

$$\iota_M : \mathfrak{t}_M \longrightarrow \mathfrak{t}_{M,1,\mathbf{w}}, \quad \iota_2 : \mathfrak{t}_2 \longrightarrow \mathfrak{t}_{M,1,\mathbf{w}}$$

such that $\iota_M(\mathfrak{t}_M) \cap \iota_2(\mathfrak{t}_2) = \{\xi_{1,\mathbf{w}}\}$. This together with (10) gives

Lemma 2.5. *There is an isomorphism of Abelian Lie algebras*

$$\mathfrak{t}_{M,1,\mathbf{w}} = \{\xi_{1,\mathbf{w}}\} + \mathfrak{h}_M + \mathfrak{h}_{\mathbf{w}}.$$

We put $\mathfrak{t}_{\mathbf{w}} = \iota_2(\mathfrak{t}_2)$ and this identifies the Sasaki cone \mathfrak{t}_2^+ of S^3 with the 2 dimensional subcone $\mathfrak{t}_{\mathbf{w}}^+$ of $\mathfrak{t}_{M,1,\mathbf{w}}^+$. Similarly, the Sasaki cone of M is identified with a subcone \mathfrak{t}_M^+ of $\mathfrak{t}_{M,1,\mathbf{w}}^+$. These two subcones intersect along the ray $\mathfrak{t}_{1,\mathbf{w}}$ generated by $\xi_{1,\mathbf{w}}$.

It is not difficult to show that

Lemma 2.6. *The \mathbf{w} -subcone $\mathfrak{t}_{\mathbf{w}}^+$ is invariant under the adjoint action of $\mathfrak{Aut}_0(M_{1,\mathbf{w}})$ on its Lie algebra.*

2.4. The Quotient Orbifolds. We are interested in Sasakian structures \mathcal{S} in the \mathbf{w} -subcone $\mathfrak{t}_{\mathbf{w}}^+$, in particular quasi-regular Sasakian structures, that is, Reeb vector fields $\xi_{\mathbf{v}}$ that lie on the integer lattice $\Lambda_{\mathbf{w}} \subset \mathfrak{t}_{\mathbf{w}}^+$. They are completely determined by the pair (v^0, v^∞) of relatively prime positive integers. This pair then determines the branch divisors D^0, D^∞ which are the zero sections and infinity sections of an orbibundle $(S_n, \Delta_{m,N}) \longrightarrow (N, \Delta_N)$ with ramification indices $\mathbf{m} = (m^0, m^\infty) = (mv^0, mv^\infty)$ through the equations $s = \gcd(l^\infty, w^0v^\infty - w^\infty v^0)$ and $l^\infty = ms$. Before giving the generalization of Theorem 3.8 of [BTF16] we describe, following [GK07], what is meant by the orbibundle

$$\pi^{orb} : (S_n, \Delta_{m,N}) \longrightarrow (N, \Delta_N).$$

First note that for $n \in \mathbb{Z}$, by S_n we mean the (total space of) the projective bundle $\mathbb{P}(\mathbb{1} \oplus L_n) \longrightarrow (N, \Delta_N)$, where $L_n \longrightarrow (N, \Delta_N)$ is an orbi-line bundle such that $c_1^{orb}(L_n) = n[\omega_N]$. Set theoretically π^{orb} is a holomorphic map $\pi : S_n \longrightarrow N$ of normal reduced complex spaces, but in the orbifold category π^{orb} pullsback the branch divisor

Δ_N to S_n giving the orbifold structure on S_n with branch divisor $\Delta_{m,N} = \Delta_{\mathbf{m}} + \pi^{-1}(\Delta_N)$ such that the generic¹ fibers of π^{orb} have the orbifold structure $\mathbb{C}\mathbb{P}^1[\mathbf{v}]/\mathbb{Z}_m = (\mathbb{C}\mathbb{P}^1, \Delta_{\mathbf{m}})$. We have the commutative diagram

$$(11) \quad \begin{array}{ccc} (S_n, \Delta_{\mathbf{m},N}) & \xrightarrow{\pi^{orb}} & (N, \Delta_N) \\ \downarrow & & \downarrow \\ (S_n, \emptyset) & \xrightarrow{\pi} & (N, \emptyset) \end{array}$$

where (N, \emptyset) is a normal projective algebraic variety N whose algebraic singularities are finite cyclic quotient singularities. So (N, \emptyset) is itself an orbifold. The vertical maps are set theoretically the identity; however, in the orbifold category they are non-trivial Galois coverings with trivial Galois group. This means that there is a non-trivial Galois covering on the level of the local uniformizing neighborhoods with a non-trivial local Galois group. Recall that a Riemannian metric and hence a Kähler metric (form) on a complex orbifold is a sequence of Kähler metrics on the local uniformizing neighborhoods that are invariant under the local uniformizing groups and that patch together by the injection maps (cf [BG08], Definition 4.2.11).

We are now ready for the generalization of Theorem 3.8 of [BTF16] to the case that M is quasi-regular.

Theorem 2.7. *Let (M, \mathcal{S}) be a quasi-regular Sasakian structure and $M_{1,\mathbf{w}} = M \star_1 S^3_{\mathbf{w}}$ the $S^3_{\mathbf{w}}$ join of M . Then the quotient of $M_{1,\mathbf{w}}$ by the S^1 action generated by a quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}_{\mathbf{w}}^+$ is the orbifold $(S_n, \Delta_{\mathbf{m}} + \pi^{-1}(\Delta_N))$ where S_n is the total space of the projective orbibundle*

$$(S_n, \Delta_{\mathbf{m}} + \pi^{-1}(\Delta_N)) \xrightarrow{\pi^{orb}} (N, \Delta_N)$$

with generic fibers $\mathbb{C}\mathbb{P}^1(\mathbf{v})/\mathbb{Z}_m$ where

$$m = \frac{l^\infty}{s}, \quad n = l^0 \frac{(w^0 v^\infty - w^\infty v^0)}{s}, \quad s = \gcd(l^\infty, |w^0 v^\infty - w^\infty v^0|)$$

and $\Delta_{\mathbf{m}}$ is a branch divisor consisting of the zero D^0 and infinity D^∞ sections of $\mathbb{P}(\mathbb{1} \oplus L_n)$ with ramification indices $\mathbf{m} = (m^0, m^\infty) = m(v^0, v^\infty)$. Moreover, $c_1^{orb}(L_n) = n[\omega_N] = n c_1^{orb}(M/N)$.

Proof. The proof given in Sections 3.3-3.5 of [BTF16] works equally well when the Sasakian structure on M is quasi-regular with only minor changes. The essential difference is that residual circle action denoted by $S^1_\theta/\mathbb{Z}_{l_2}$ in [BTF16] is now only locally free. As in [BTF16] the commutative diagram (5) gives the commutative diagram

$$(12) \quad \begin{array}{ccc} M \times L(l^\infty; l^0 w^0, l^0 w^\infty) & & \\ & \searrow \pi_L & \\ & & M_{1,\mathbf{w}} \\ & & \swarrow \pi_{\mathbf{v}} \\ & & M_{1,\mathbf{w}}/S^1_\phi \\ & \downarrow \pi_{\mathbb{T}^2} & \end{array}$$

¹By generic fiber we mean the fiber over a point in the orbifold regular locus of N .

where $L(l^\infty; l^0 w^0, l^0 w^\infty)$ is a lens space. The action of $\mathbb{T}^2 = S_\phi^1 \times (S_\theta^1/\mathbb{Z}_{l^\infty})$ on $M \times L(l^\infty; l^0 w^0, l^0 w^\infty)$ is given by

$$(13) \quad (u; z_1, z_2) \mapsto (\mathcal{A}(\theta, u); [e^{i(v^0 \phi - \frac{l^0 w^0}{l^\infty} \theta)} z_1, e^{i(v^\infty \phi - \frac{l^0 w^\infty}{l^\infty} \theta)} z_2]),$$

where now the action \mathcal{A} on M is only locally free. This has two main consequences. First, $M_{1, \mathbf{w}}$ is a Sasaki orbifold unless Equation (9) holds in which case $M_{1, \mathbf{w}}$ will be a smooth Sasaki manifold. Second, and more importantly, the quotient $M_{1, \mathbf{w}}/S_\theta^1/\mathbb{Z}_{l^\infty}$ can have an additional branch divisor which is the inverse image $\pi^{-1}(\Delta_N)$. This gives branch divisors on $(S_n, \Delta_{N, \mathbf{m}})$ as

$$(14) \quad \Delta_{N, \mathbf{m}} = (1 - \frac{1}{m^0})D^0 + (1 - \frac{1}{m^\infty})D^\infty + \sum_{j=1}^k (1 - \frac{1}{m_j})\pi^{-1}(D_j).$$

As is well understood this shifts the orbifold first Chern class. Other than this, the argument is the same. So the result follows as in [BTF16]. \square

Remark 2.8. Note that for m and n as in Theorem 2.7 we have $\gcd(m, n) = 1$. This is due to the fact that $\gcd(l^0, l^\infty) = 1$, any factor of m is a factor of l^∞ , and, by definition of s , $\gcd(m, \frac{w^0 v^\infty - w^\infty v^0}{s}) = 1$.

Generally we have

Lemma 2.9. *Let $M_{1, \mathbf{w}}$ be the $S_{\mathbf{w}}^3$ -join with a quasiregular Sasaki manifold $M \rightarrow N$ of order Υ_N . Let $\mathcal{S}_{\mathbf{v}} = (\xi_{\mathbf{v}}, \eta_{\mathbf{v}}, \Phi, g_{\mathbf{v}})$ be a quasiregular Sasakian structure in the \mathbf{w} cone $\mathfrak{t}_{\mathbf{w}}^+$. The order Υ of $\mathcal{S}_{\mathbf{v}}$ is the product $mv^0 v^\infty \Upsilon_N$.*

Proof. The local uniformizing groups of the orbifold are just the isotropy subgroups of the \mathbb{T}^2 action (13), and by definition the order of an orbifold is the least common multiple of the orders of the isotropy groups. The orbifold in question is the quotient of $M \times L(l^\infty; l^0 w^0, l^0 w^\infty)$ by the \mathbb{T}^2 action (13) which is free on the dense open subset

$$\mathcal{Z} = \left(M \setminus \pi_M^{-1} \circ \pi^{-1}(\Delta_N) \right) \times \left(L(l^\infty; l^0 w^0, l^0 w^\infty) \setminus \{[z_1 = 0] \cup [z_2 = 0]\} \right)$$

where $|z_1|^2 + |z_2|^2 = 1$ is understood. Let $x \in N$ be a point with isotropy group \mathbb{Z}_r and $y \in [z_2 = 0]$. Then the isotropy group at (x, y) is $\mathbb{Z}_r \times \mathbb{Z}_{m^0}$. Similarly, if $y \in [z_1 = 0]$ the isotropy group at (x, y) is $\mathbb{Z}_r \times \mathbb{Z}_{m^\infty}$. Running through the orbifold singular set of N gives the order $\Upsilon_N \text{lcm}(m^0, m^\infty) = \Upsilon_N m v^0 v^\infty$. \square

2.5. The Transverse Kähler Structure. It turns out that as in Section 6.1 of [BTF16] we can lift the Kähler data on the orbifolds $(S_n, \Delta_{\mathbf{m}, N})$ to the transverse Kähler data on a quasi-regular Sasaki manifold. Since quasi-regular Reeb fields are dense in $\mathfrak{t}_{\mathbf{w}}^+$ we can extend this transverse data to the entire \mathbf{w} -cone and as in [BTF16] (27) gives a smooth family of Sasakian structures on the join $M_{1, \mathbf{w}}$. In the following we shall describe the details.

2.5.1. Generalized orbifold Calabi construction. The generalized Calabi construction was presented in [ACGTF04, Section 2.5] and further discussed in [ACGTF11, Section 2.3]. Here we make a modest generalization, allowing for the base space to be a normal projective algebraic variety with cyclic orbifold singularities. We first discussed this type of generalization in [BTF19] for the purpose of constructing certain explicit

Sasaki-Einstein metrics on a class of 7-manifolds, but here we will not restrict the base to be a Hirzebruch orbifold.

The ingredients in the construction are as follows:

- [Base] A Kähler orbifold (N, Δ_N) equipped with a Kähler orbifold metric, g_{base} , whose Kähler form, $\omega_{base} = 2\pi\omega_N$, satisfies that $[\omega_N]$ is a primitive orbifold Kähler class.
- [Fiber] A weighted projective line $(\mathbb{C}\mathbb{P}^1_{m^0, m^\infty} = \mathbb{C}\mathbb{P}^1_{v^0, v^\infty}/\mathbb{Z}_m, g_{\mathbf{m}}, \omega_{\mathbf{m}})$ with orbifold Kähler structure $(g_{\mathbf{m}}, \omega_{\mathbf{m}})$ and rational Delzant polytope $[-1, 1] \subseteq \mathbb{R}^*$ and momentum map $\mathfrak{z}: \mathbb{C}\mathbb{P}^1_{m^0, m^\infty} \rightarrow [-1, 1]$. Here $(m^0, m^\infty) = m(v^0, v^\infty)$ and v^0, v^∞ are coprime.
- A principal S^1 orbi-bundle, $P_n \rightarrow (N, \Delta_N)$, with a principal connection of curvature $n\omega_{base} \in \Omega^{1,1}((N, \Delta_N), \mathbb{R})$, where S^1 acts on $\mathbb{C}\mathbb{P}^1_{m^0, m^\infty}$, $n \in \mathbb{Z} \setminus \{0\}$, and $\gcd(n, m) = 1$. Note that $n \in \text{span}_{\mathbb{Z}}\{v^0, v^\infty\}$ (since v^0, v^∞ are coprime), so $mn \in \text{span}_{\mathbb{Z}}\{m^0, m^\infty\}$.
- A constant $0 < |r| < 1$ with the same sign as n [ensuring that the $(1, 1)$ -form $(1/r + \mathfrak{z})n\omega_{base}$ is positive for $\mathfrak{z} \in [-1, 1]$].

From this data we may define the orbifold

$$(S_n, \Delta_{\mathbf{m}, N}) = P_n \times_{S^1} \mathbb{C}\mathbb{P}^1_{m^0, m^\infty} = \mathring{M} \times_{\mathbb{C}^*} \mathbb{C}\mathbb{P}^1_{m^0, m^\infty} \rightarrow (N, \Delta_N),$$

where $\mathring{S}_n = P_n \times_{S^1} (\mathfrak{z}^{-1}(-1, 1))$. Since the curvature 2-form of P_n has type $(1, 1)$, \mathring{S}_n is a holomorphic principal \mathbb{C}^* bundle with connection $\theta \in \Omega^1(\mathring{S}_n, \mathbb{R})$ and $(S_n, \Delta_{\mathbf{m}, N})$ is a complex orbifold.

On \mathring{S}_n we define Kähler structures of the form

$$(15) \quad \begin{aligned} g &= (1/r + \mathfrak{z})n (\pi^{orb})^* g_{base} + \frac{1}{\Theta(\mathfrak{z})} d\mathfrak{z}^2 + \Theta(\mathfrak{z})\theta^2 \\ \omega &= (1/r + \mathfrak{z})n (\pi^{orb})^* \omega_{base} + d\mathfrak{z} \wedge \theta \\ d\theta &= n (\pi^{orb})^* \omega_{base}, \end{aligned}$$

where $\frac{1}{\Theta(\mathfrak{z})} = \frac{d^2U}{d\mathfrak{z}^2}$ and U is the symplectic potential [Gui94] of the chosen toric Kähler structure $g_{\mathbf{m}}$ on $\mathbb{C}\mathbb{P}^1_{m^0, m^\infty}$.

The *generalized Calabi construction* arises from seeing (15) as a blueprint for the construction of various orbifold Kähler metrics on $(S_n, \Delta_{\mathbf{m}, N})$ by choosing various smooth functions $\Theta(\mathfrak{z})$ on $(-1, 1)$ satisfying that

$$(16) \quad \begin{aligned} &\bullet \text{ [boundary values] the following endpoint conditions are satisfied} \\ \Theta(\pm 1) &= 0 \quad \text{and} \quad \Theta'(-1) = 2/m^\infty \quad \text{and} \quad \Theta'(1) = -2/m^0; \end{aligned}$$

- [positivity] the function $\Theta(\mathfrak{z})$ is positive for $\mathfrak{z} \in (-1, 1)$.

Metrics constructed this way are called *compatible Kähler metrics* with *compatible Kähler classes* parametrized by r . If g_{base} has constant curvature we call the compatible metrics/classes *admissible*. Moving forward we will assume to be in that case.

Note that due to [ACG06] (see also (46) of [BTF16]), we have that if $g_{base} = 2\pi g_N$ has complex dimension d_N and constant scalar curvature $2d_N A_N$, then on \mathring{S}_n the scalar

curvature of the admissible metric in (15) is given by

$$(17) \quad Scal = \frac{2d_N A_N r}{n(1+r\mathfrak{z})} - \frac{F''(\mathfrak{z})}{(1+r\mathfrak{z})^{d_N}},$$

where $F(\mathfrak{z}) = (1+r\mathfrak{z})^{d_N} \Theta(\mathfrak{z})$. [Note that “ A_N ” plays the same role as “ A ” introduced in Section 6.2 of [BTF16].]

Remark 2.10. Note that the Calabi toric metrics in [Leg11b] are special cases of the above ansatz. Such orbifold constructions also appear explicitly in Sections 3.5 and 4.4.2 of [ACGL20].

On each uniformizing cover of $(S_n, \Delta_{\mathbf{m},N})$ we have (following Section 1.3 of [ACGTF08]) that

$$\phi^*[\omega] = 2\pi \frac{n}{r} \phi^*(\pi^{orb})^*[\omega_N] + 2\pi(\phi^*PD(D^0) + \phi^*PD(D^\infty)).$$

Since $\phi^*PD(D^0) - \phi^*PD(D^\infty) = n\phi^*(\pi^{orb})^*[\omega_N]$, this may be written as

$$\phi^*[\omega] = 2\pi\left(\frac{n}{r} + n\right)\phi^*(\pi^{orb})^*[\omega_N] + 4\pi\phi^*PD(D^\infty)$$

or

$$(18) \quad [\omega] = 4\pi \left(\frac{n(1+r)}{2r} (\pi^{orb})^*[\omega_N] + PD(D^\infty) \right).$$

2.5.2. *The transverse Kähler structure.* First note that a primitive orbifold class $[\omega_N]$ is obtained from a primitive integer class $[\omega_N]_I$, viz

$$(19) \quad [\omega_N] = \frac{[\omega_N]_I}{\Upsilon_N}.$$

Lemma 2.11. *The induced primitive orbifold Kähler form $\omega_{n,\mathbf{m},N}$ on the orbifold $(S_n, \Delta_{N,\mathbf{m}})$ satisfies*

$$[\omega_{n,\mathbf{m},N}] = \frac{l^0 w^0 v^\infty (\pi^{orb})^*[\omega_N]_I + s\Upsilon_N PD(D^\infty)}{\gcd(s\Upsilon_N, w^0 v^\infty l^0) m v^0 v^\infty \Upsilon_N}$$

where $[\omega_N]_I$ is a primitive integer class.

Proof. The analogue of Diagram (35) in [BTF16] is

$$(20) \quad \begin{array}{ccc} M \times L(l^\infty; l^0 w^0, l^0 w^\infty) & & \\ \downarrow \pi_L & & \\ M_{1,\mathbf{w}} & & \\ \swarrow p_{\mathbf{w}} & & \searrow p_{\mathbf{v}} \\ (N, \Delta_N) \times \mathbb{CP}^1[\mathbf{w}] & & (S_n, \Delta_{\mathbf{m},N}) \\ \swarrow pr_1 & & \swarrow \pi^{orb} \\ & (N, \Delta_N) & \end{array}$$

where $p_{\mathbf{w}}, p_{\mathbf{v}}, pr_1, \pi^{orb}$ are the obvious projections. The fibers of the map π^{orb} are orbifolds of the form $\mathbb{CP}^1[\mathbf{v}]/\mathbb{Z}_m$. As is [BTF16] and by using the diagram (20) we

have that there is a generator $\gamma \in H^2(M_{\mathbf{1},\mathbf{w}}, \mathbb{Z})$ such that $(pr_1 \circ p_{\mathbf{w}})^*[\omega_N] = l^\infty \gamma$ and $(pr_2 \circ p_{\mathbf{w}})^*[\omega_{\mathbf{w}}] = -l^0 \gamma$. Now (again, similarly to [BTF16]), using Lemma 2.16 together with the fact that

$$c_1^{orb}((N, \Delta_N) \times \mathbb{CP}^1[\mathbf{w}]) = pr_1^* c_1^{orb}(N, \Delta_N) + |\mathbf{w}| pr_2^*[\omega_{\mathbf{w}}],$$

and that, as the diagram (20) suggests, both pull-backs to $M_{\mathbf{1},\mathbf{w}}$ of the orbifold Chern class $c_1^{orb}(N, \Delta_N)$ should agree, we arrive at the equation

$$\frac{1}{m^0} p_{\mathbf{v}}^* PD(D^0) + \frac{1}{m^\infty} p_{\mathbf{v}}^* PD(D^\infty) = -l^0 |\mathbf{w}| \gamma.$$

On each uniformizing cover we also have that

$$\varphi^* PD(D^0) - \varphi^* PD(D^\infty) = n \varphi^* (\pi_{\mathbf{v}}^{orb})^* [\omega_N]$$

and hence using $(\pi^{orb} \circ p_{\mathbf{v}})^*[\omega_N] = (pr_1 \circ p_{\mathbf{w}})^*[\omega_N] = l^\infty \gamma$ we end up with the system

$$(21) \quad p_{\mathbf{v}}^* PD(D^0) - p_{\mathbf{v}}^* PD(D^\infty) = l^\infty n \gamma,$$

$$(22) \quad \frac{1}{m^0} p_{\mathbf{v}}^* PD(D^0) + \frac{1}{m^\infty} p_{\mathbf{v}}^* PD(D^\infty) = -l^0 |\mathbf{w}| \gamma.$$

As in [BTF16] the solution to this system is

$$(23) \quad p_{\mathbf{v}}^* PD(D^\infty) = -m w^0 v^\infty l^0 \gamma$$

$$(24) \quad p_{\mathbf{v}}^* PD(D^0) = -m w^\infty v^0 l^0 \gamma$$

We keep diagram (20) in mind. Now the induced orbifold Kähler class on $(S_n, \Delta_{N,\mathbf{m}})$ is in the kernel of $p_{\mathbf{v}}^*$, and as in the proof of Lemma 3.10 in [BTF16] both $\{(\pi^{orb})^*[\omega_N], PD(D^0)\}$ and $\{(\pi^{orb})^*[\omega_N], PD(D^\infty)\}$ span $\ker(p_{\mathbf{v}} \circ \pi_L)^*$. Now we can write a primitive orbifold Kähler form $\omega_{n,\mathbf{m},N}$ as a primitive integer Kähler form divided by the order Υ of the orbifold. So on $(S_n, \Delta_{N,\mathbf{m}})$ using Lemma 2.9 we write

$$[\omega_{n,\mathbf{m},N}] = \frac{k_1 (\pi^{orb})^*[\omega_N]_I + k_2 PD(D^\infty)}{m v^0 v^\infty \Upsilon_N}$$

for some positive integers k_1, k_2 . Since $[\omega_{n,\mathbf{m},N}]$ is in the kernel of $p_{\mathbf{v}}^*$ we have

$$k_1 \Upsilon_N l^\infty - k_2 m w^0 v^\infty l^0 = 0.$$

This determines $k_1/k_2 = m w^0 v^\infty l^0 / (l^\infty \Upsilon_N) = m w^0 v^\infty l^0 / (m s \Upsilon_N) = w^0 v^\infty l^0 / (s \Upsilon_N)$. Ensuring primitivity then gives the result. \square

Lemma 2.12. *The induced primitive orbifold Kähler form $\omega_{n,\mathbf{m},N}$ on the orbifold $(S_n, \Delta_{N,\mathbf{m}})$ satisfies*

$$[\omega_{n,\mathbf{m},N}] = \frac{s}{4\pi \gcd(s \Upsilon_N, w^0 v^\infty l^0) m v^0 v^\infty} [\omega],$$

where $[\omega]$ is the admissible Kähler class from (18) with $r = \frac{w^0 v^\infty - w^\infty v^0}{w^0 v^\infty + w^\infty v^0}$.

Proof. From (18) and (19) we have that

$$\begin{aligned}
[\omega] &= 4\pi \left(\frac{n(1+r)}{2r} \frac{(\pi^{orb})^*[\omega_N]_I}{\Upsilon_N} + PD(D^\infty) \right) \\
&= \frac{4\pi}{s\Upsilon_N} \left(\frac{sn(1+r)}{2r} (\pi^{orb})^*[\omega_N]_I + s\Upsilon_N PD(D^\infty) \right) \\
&= \frac{4\pi}{s\Upsilon_N} \left(\frac{l^0(w^0v^\infty - w^\infty v^0)(1+r)}{2r} (\pi^{orb})^*[\omega_N]_I + s\Upsilon_N PD(D^\infty) \right).
\end{aligned}$$

This is a multiple of $[\omega_{n,\mathbf{m},N}]$ iff $r = \frac{w^0v^\infty - w^\infty v^0}{w^0v^\infty + w^\infty v^0}$ and with this value for r , we get

$$\begin{aligned}
[\omega] &= \frac{4\pi}{s\Upsilon_N} \left(\frac{l^0(w^0v^\infty - w^\infty v^0)(1+r)}{2r} (\pi^{orb})^*[\omega_N]_I + s\Upsilon_N PD(D^\infty) \right) \\
&= \frac{4\pi}{s\Upsilon_N} (l^0 w^0 v^\infty (\pi^{orb})^*[\omega_N]_I + s\Upsilon_N PD(D^\infty)) \\
&= \frac{4\pi \gcd(s\Upsilon_N, w^0v^\infty l^0) m v^0 v^\infty}{s} \frac{(l^0 w^0 v^\infty (\pi^{orb})^*[\omega_N]_I + s\Upsilon_N PD(D^\infty))}{\gcd(s\Upsilon_N, w^0v^\infty l^0) m v^0 v^\infty \Upsilon_N} \\
&= \frac{4\pi \gcd(s\Upsilon_N, w^0v^\infty l^0) m v^0 v^\infty}{s} [\omega_{n,\mathbf{m},N}].
\end{aligned}$$

The result now follows. \square

Note that since $\omega_{n,\mathbf{m},N}$ is the induced primitive orbifold Kähler form on $(S_n, \Delta_{N,\mathbf{m}})$, we could write

$$[\omega_{n,\mathbf{m},N}] = \frac{[\omega_{n,\mathbf{m},N}]_I}{m v^0 v^\infty \Upsilon_N},$$

where

$$[\omega_{n,\mathbf{m},N}]_I = \frac{l^0 w^0 v^\infty (\pi^{orb})^*[\omega_N]_I + s\Upsilon_N PD(D^\infty)}{\gcd(s\Upsilon_N, w^0v^\infty l^0)}.$$

Using this notation we can formulate the equation in Lemma 2.12 as

$$[\omega_{n,\mathbf{m},N}]_I = \frac{s\Upsilon_N}{4\pi \gcd(s\Upsilon_N, w^0v^\infty l^0)} [\omega].$$

Lemma 2.12 tells us that the class of the induced primitive orbifold Kähler form $\omega_{n,\mathbf{m},N}$ on the orbifold $(S_n, \Delta_{N,\mathbf{m}})$ is a constant multiple of the admissible Kähler class $[\omega]$ with the prescribed value of r . Since $(\pi^{orb})^*\omega_{n,\mathbf{m},N}$ is in turn a constant multiple of $d\eta_{\mathbf{v}}$, this means that for the admissible Kähler form, we may say that (up to isotopy)

$$(25) \quad p_{\mathbf{v}}^* \omega = b d\eta_{\mathbf{v}}$$

for some positive constant b . To determine the value of b we can follow the proof of Proposition 6.2 of [BTF16]. Since our notation here is slightly different we will summarize the argument in broad strokes:

As in [BTF16] we have the moment map of the lifted circle action of the moment map $\mathfrak{z}, \tilde{\mathfrak{z}} : M_{1,\mathbf{w}} \rightarrow [-1, 1]$ and

$$(26) \quad d\eta_{\mathbf{v}}|_{\mathcal{D}} = \frac{w^0v^\infty - w^\infty v^0}{2v^0v^\infty} (\tilde{\mathfrak{z}} + r^{-1}) d\eta_{\mathbf{w}}|_{\mathcal{D}}.$$

Moreover, identifying N with the zero section of S_n , $\omega|_N = 2\pi n(\tilde{\mathfrak{z}} + r^{-1})\omega_N$. Now comparing coefficients of the pullback of ω_N on both sides of equation (25) and using

equation (26) with the commutative diagram (20) (giving that the two pull-backs of ω_N should agree), we get the equation

$$2\pi n = b \frac{w^0 v^\infty - w^\infty v^0}{2v^0 v^\infty} l^0.$$

Using the value of n from Theorem 2.7 we arrive at $b = \frac{4\pi m v^0 v^\infty}{l^\infty}$ and thus we have that - up to isotopy - the transverse Kähler structure (g^T, ω^T) satisfies

$$\omega^T = d\eta_{\mathbf{v}} = \frac{l^\infty}{4\pi} \frac{p_{\mathbf{v}}^* \omega}{m v^0 v^\infty}$$

and

$$g^T = \frac{l^\infty}{4\pi} \frac{p_{\mathbf{v}}^* g}{m v^0 v^\infty}.$$

Thus, similarly to Section 6 of [BTF16] we can lift the admissible data to $M_{1,\mathbf{w}}$ and we have the following theorem:

Theorem 2.13. *The transverse Kähler metric and Kähler form are (possibly up to isotopy) given by*

$$(27) \quad \begin{aligned} g^T &= \frac{l^\infty}{4\pi} \left(\frac{2\pi l^0 (w^0 v^\infty - w^\infty v^0)}{l^\infty v^0 v^\infty} (r^{-1} + \tilde{\mathfrak{J}}) (\pi^{orb} \circ p_{\mathbf{v}})^* g_N + \frac{d\tilde{\mathfrak{J}}^2}{\tilde{\Theta}(\tilde{\mathfrak{J}})} + \tilde{\Theta}(\tilde{\mathfrak{J}}) \tilde{\theta}^2 \right) \\ \omega^T &= \frac{l^\infty}{4\pi} \left(\frac{2\pi l^0 (w^0 v^\infty - w^\infty v^0)}{l^\infty v^0 v^\infty} (r^{-1} + \tilde{\mathfrak{J}}) (\pi^{orb} \circ p_{\mathbf{v}})^* \omega_N + d\tilde{\mathfrak{J}} \wedge \tilde{\theta} \right) \end{aligned}$$

where

$$r = \frac{w^0 v^\infty - w^\infty v^0}{w^0 v^\infty + w^\infty v^0}, \quad \tilde{\Theta} = m v^0 v^\infty p_{\mathbf{v}}^* \Theta, \quad \tilde{\theta} = \frac{p_{\mathbf{v}}^* \theta}{m v^0 v^\infty},$$

and that $\tilde{\Theta}$ satisfies the boundary conditions $\tilde{\Theta}(\pm 1) = 0$, $\tilde{\Theta}'(-1) = 2v^0$ and $\tilde{\Theta}'(1) = -2v^\infty$.

The quasi-regular transverse Kähler structure extends smoothly to the full \mathbf{w} Sasaki cone $\mathfrak{t}_{\mathbf{w}}^+$ and converges to the Kähler forms of the reducible Sasakian structure [BHLTF18] as $\mathbf{v} \rightarrow \mathbf{w}$.

It is clear from the explicit forms (27) that they extend smoothly to the full \mathbf{w} Sasaki cone $\mathfrak{t}_{\mathbf{w}}^+$. And since

$$\frac{l^\infty}{4\pi} \frac{2\pi l^0 (w^0 v^\infty - w^\infty v^0)}{l^\infty v^0 v^\infty} r^{-1} \longrightarrow l^0$$

we see that as $\mathbf{v} \rightarrow \mathbf{w}$, the Sasakian structure converges to the reducible Sasakian structure defined by the join. So the transverse Kähler structure in Theorem 2.13 extends smoothly to the full \mathbf{w} Sasaki cone $\mathfrak{t}_{\mathbf{w}}^+$ and converges to the Kähler forms of the reducible Sasakian structure [BHLTF18] as $\mathbf{v} \rightarrow \mathbf{w}$.

By combining Lemma 2.12 and Theorem 2.13, we have that

$$\omega^T = d\eta_{\mathbf{v}} = \frac{l^\infty}{4\pi} \frac{4\pi \gcd(s\Upsilon_N, w^0 v^\infty l^0) m v^0 v^\infty}{s} p_{\mathbf{v}}^* \omega_{n,\mathbf{m},N},$$

which gives the corollary below.

Corollary 2.14. *The transverse Kähler structure (27) satisfies*

$$(28) \quad \omega^T = d\eta_{\mathbf{v}} = m \gcd(s\Upsilon_N, w^0 v^\infty l^0) p_{\mathbf{v}}^* \omega_{n,\mathbf{m},N}.$$

Remark 2.15. We see, as discussed in Remark 2.2, the quotient orbifold $(S_n, \Delta_{\mathbf{m}, N})$ of the primitive Sasakian structure on $M_{1, \mathbf{w}}$ is not necessarily primitive.

2.6. Extremal, CSC, and KE admissible metrics on $(S_n, \Delta_{\mathbf{m}, N})$. Completely similar to Sections 5.1-5.3 of [BTF16] (which is turn is based on the admissible construction of Apostolov, Calderbank, Gauduchon, and Tønnesen-Friedman [ACG06, ACGTF04, ACGTF08]) we may now impose geometric conditions on the metric g in (15). Using $F(\mathfrak{z}) = (1 + r\mathfrak{z})^{d_N} \Theta(\mathfrak{z})$ as in (17) above, we then have

- g is an **extremal Kähler metric** with $Scal = -(\alpha\mathfrak{z} + \beta)$ if and only if

$$F''(\mathfrak{z}) = (1 + r\mathfrak{z})^{d_N-1} \left(\frac{2d_N A_N r}{n} + (\alpha\mathfrak{z} + \beta)(1 + r\mathfrak{z}) \right).$$

By integration, there is a unique solution $F_{ext}(\mathfrak{z})$ (the *extremal polynomial*) to this boundary value problem, where the constants α and β are such that the endpoint conditions of (16) are satisfied. Further, $F_{ext}(\mathfrak{z})$ corresponds to a genuine orbifold Kähler metric on $(S_n, \Delta_{\mathbf{m}, N})$ if $F_{ext}(\mathfrak{z}) > 0$ for $-1 < \mathfrak{z} < 1$. As in Section 5.2 of [BTF16] and the references therein, we can see that this is the case if $A_N \geq 0$.

- The extremal polynomial $F_{ext}(\mathfrak{z})$ above corresponds to a **CSC Kähler metric** with $Scal = -\beta$ if and only if ($\alpha = 0$ which is equivalent to)

$$(29) \quad \frac{2A_N ((1+r)^{d_N+1} - (1-r)^{d_N+1})}{nr(d_N+1)} + \frac{\beta ((1+r)^{d_N+2} - (1-r)^{d_N+2})}{r^2(d_N+1)(d_N+2)} + 2c = 0,$$

where

$$\beta = \frac{-2(d_N+1)r (m^\infty(1+r)^{d_N}(n+m^0A_N) - m^0(1-r)^{d_N}(-n+m^\infty A_N))}{nm^0m^\infty ((1+r)^{d_N+1} - (1-r)^{d_N+1})}$$

and²

$$c = \frac{2(1-r^2)^{d_N} (nm^\infty(1-r) + nm^0(1+r) - 2m^0m^\infty A_N)}{nm^0m^\infty ((1+r)^{d_N+1} - (1-r)^{d_N+1})}.$$

[Note that in this case, $F_{ext}(\mathfrak{z}) > 0$ for $-1 < \mathfrak{z} < 1$ is automatic.]

- Finally, the extremal polynomial $F_{ext}(\mathfrak{z})$ above corresponds to a **KE Kähler metric** if and only if (N, ω_N, g_N) is KE with index \mathcal{J}_N (note $A_N = \mathcal{J}_N$) and

$$(30) \quad \int_{-1}^1 ((1-\mathfrak{z})/m^\infty - (1+\mathfrak{z})/m^0) (1+r\mathfrak{z})^{d_N} d\mathfrak{z} = 0$$

and

$$2r\mathcal{J}_N/n = (1+r)/m^\infty + (1-r)/m^0.$$

[Again, in this case, $F_{ext}(\mathfrak{z}) > 0$ for $-1 < \mathfrak{z} < 1$ is automatic.]

²There is an inconsequential r missing in the last term in the numerator of Equation (48) in [BTF16]

2.7. The Orbifold First Chern Class. We also have the analogue of Lemma 3.9 of [BTF16] when N is an orbifold, viz.

Lemma 2.16. *For the projective orbibundle*

$$(S_n, \Delta_{\mathbf{m}, N}) \xrightarrow{\pi^{orb}} (N, \Delta_N)$$

we have

$$c_1^{orb}(S_n, \Delta_{\mathbf{m}, N}) = (\pi_{\mathbf{m}}^{orb})^* c_1^{orb}(N, \Delta_N) + \frac{1}{m^0} PD(D^0) + \frac{1}{m^\infty} PD(D^\infty).$$

Proof. For the orbifold (N, Δ_N) the orbifold canonical divisor K_N^{orb} satisfies

$$K_N^{orb} = \varphi^* K_N + \sum_j \left(1 - \frac{1}{m_j}\right) \varphi^* D_j$$

where φ is local uniformizing map whose notation we often omit. Taking the Poincaré dual and using $c_1(N) = -c_1(K_N)$ then gives

$$(31) \quad c_1^{orb}(N, \Delta_N) = \varphi^* c_1(N) + \sum_j \left(\frac{1}{m_j} - 1\right) \varphi^* PD(D_j).$$

Now for a projective bundle $\mathbb{P}(\mathbb{1} \oplus L_n) \xrightarrow{\pi} N$ we know from Leray-Hirsch that the cohomology $H^*(\mathbb{P}(\mathbb{1} \oplus L_n), \mathbb{Z})$ is a free module over $H^*(N, \mathbb{Z})$ with basis $\{1, x\}$ where x is the cohomology class in $H^2(\mathbb{P}(\mathbb{1} \oplus L_n), \mathbb{Z})$ whose restriction to a fiber $\approx \mathbb{C}\mathbb{P}^1$ freely generates the cohomology of $\mathbb{C}\mathbb{P}^1$. It is the first Chern class of the dual of the tautological bundle [BT82]. So the vertical bundle $\mathcal{V}(N)$ has first Chern class $PD(D^0) + PD(D^\infty)$ which implies

$$c_1(\mathbb{P}(\mathbb{1} \oplus L_n)) = \pi^* c_1(N) + PD(D^0) + PD(D^\infty).$$

This together with (31) gives the result. \square

Remark 2.17. A choice of quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}_{\mathbf{w}}^+$ completely determines the quotient orbifold structure $(S_n, \Delta_{\mathbf{m}, N})$. With the join parameters $\mathbf{l} = (l^0, l^\infty)$ and $\mathbf{w} = (w^0, w^\infty)$ fixed, the Reeb vector field $\xi_{\mathbf{v}}$ uniquely determines the integers n, m, s by the expressions in Theorem 2.7. Note that both m and s are divisors of l^∞ .

Remark 2.18. From the join construction we get a projective orbibundle

$$\mathbb{P}(\mathbb{1} \oplus L_n) = (S_n, \Delta_{N, \mathbf{m}}) \xrightarrow{\pi} (N, \Delta_N)$$

with generic fibers of the form $\mathbb{C}\mathbb{P}^1[\mathbf{v}]/\mathbb{Z}_m$. However, now since (N, Δ_N) is also an orbifold we have singular fibers along the branch divisors D_j of N and the order of the orbifold singularity is given by the ramification index m_j . So along D_j the fibers of π have the form $\mathbb{C}\mathbb{P}^1[\mathbf{v}]/(\mathbb{Z}_m \times \mathbb{Z}_{m_j})$, and along intersections $D_i \cap D_j$ with $i \neq j$ the fibers take the form $\mathbb{C}\mathbb{P}^1[\mathbf{v}]/(\mathbb{Z}_m \times \mathbb{Z}_{m_i} \times \mathbb{Z}_{m_j})$, etc.

Remark 2.19. In the regular case where $\Upsilon_N = 1$ we have

$$[\omega_{n, \mathbf{m}, N}] = \frac{l^0 v^\infty w^0 (\pi_{\mathbf{v}}^{orb})^* [\omega_N] + s PD(D^\infty)}{\gcd(s, w^0 v^\infty l^0) m v^0 v^\infty}$$

as expected from Lemma 3.11 in [BTF16] and the assumption of primitivity.

Example 2.1. If we assume as in [BTF19] that (N, Δ_N) is KE with $c_1^{orb}(N, \Delta_N) = \mathcal{J}_N[\omega_N]$ and l^0, l^∞ are chosen such that

$$(32) \quad l^0 = \frac{\mathcal{J}_N}{\gcd(w^0 + w^\infty, \mathcal{J}_N)}, \quad l^\infty = \frac{w^0 + w^\infty}{\gcd(w^0 + w^\infty, \mathcal{J}_N)},$$

then from Lemma 2.16 we calculate that

$$\begin{aligned} c_1^{orb}(S_n, \Delta_{\mathbf{m}, N}) &= (\pi^{orb})^* c_1^{orb}(N, \Delta_N) + \frac{1}{m^0} PD(D^0) + \frac{1}{m^\infty} PD(D^\infty) \\ &= \frac{v^\infty(\mathcal{J}_N m v^0 + n)(\pi^{orb})^*[\omega_N] + (v^0 + v^\infty) PD(D^\infty)}{m v^0 v^\infty} \\ &= \frac{(l^\infty \mathcal{J}_N v^0 v^\infty + l^0 v^\infty (w^0 v^\infty - w^\infty v^0))(\pi_{\mathbf{m}}^{orb})^*[\omega_N] + s(v^0 + v^\infty) PD(D^\infty)}{s m v^0 v^\infty} \\ &= \frac{((w^0 + w^\infty) v^0 v^\infty + v^\infty (w^0 v^\infty - w^\infty v^0)) l^0 (\pi^{orb})^*[\omega_N] + s(v^0 + v^\infty) PD(D^\infty)}{s m v^0 v^\infty} \\ &= \frac{(v^0 + v^\infty)(w^0 v^\infty l^0 (\pi^{orb})^*[\omega_N]_I + s \Upsilon_N PD(D^\infty))}{s m v^0 v^\infty \Upsilon_N} \\ &= \left(\frac{v^0 + v^\infty}{s} \right) \gcd(l^0 w^0 v^\infty, s \Upsilon_N)[\omega_{n, \mathbf{m}, N}] \end{aligned}$$

which implies that the Fano index of $[\omega_{n, \mathbf{m}, N}]$ is

$$\mathcal{J}_{(S_n, \Delta_{\mathbf{m}, N})} = \left(\frac{v^0 + v^\infty}{s} \right) \gcd(l^0 w^0 v^\infty, s \Upsilon_N).$$

Now it follows from first principles that the index $\mathcal{J}_{(S_n, \Delta_{\mathbf{m}, N})}$ must be an integer. But we can show this directly in our case. First note that since s divides $|\mathbf{w}| = w^0 + w^\infty$ when $c_1(\mathcal{D}_{\mathbf{1}, \mathbf{w}}) = 0$ and it also divides $(w^0 v^\infty - w^\infty v^0)$, it is a factor of $v^0(w^0 + w^\infty) + (w^0 v^\infty - w^\infty v^0) = w^0(v^0 + v^\infty)$. Similarly, it is a factor of $v^\infty(w^0 + w^\infty) - (w^0 v^\infty - w^\infty v^0) = w^\infty(v^0 + v^\infty)$. So, since $\gcd(w^0, w^\infty) = 1$, it must be a factor of $(v^0 + v^\infty)$. A similar analysis shows that $\gcd(l^0 w^0 v^\infty, s) = 1$. So the index reduces to

$$(33) \quad \mathcal{J}_{(S_n, \Delta_{\mathbf{m}, N})} = \left(\frac{v^0 + v^\infty}{s} \right) \gcd(l^0 w^0 v^\infty, \Upsilon_N).$$

As a concrete example, using (33), the index of the quotient Kähler-Einstein orbifold coming from the quasi-regular Sasaki-Einstein metric on $Y^{13,8}$ is determined to be $\mathcal{J}_{v_2} = 12$ as in agreement with the first example of Example 5.3 of [BTF19]. Continuing with this example we can now consider the quasi-regular Sasaki-Einstein metric of the join

$$Y^{13,8} \star_{4,15} S_{34,11}^3$$

and from the data given in [BTF19], (33) tells us that now the index of the quotient is $\mathcal{J}_{v_3} = 28$.

There are obstructions to the regularity of Reeb vector fields:

Proposition 2.20. *Let $M_{\mathbf{1}, \mathbf{w}}$ be a Sasaki join manifold with $l^\infty > 2$ and $c_1(\mathcal{D}_{\mathbf{1}, \mathbf{w}}) = 0$. Then the \mathbf{w} cone $\mathfrak{t}_{\mathbf{w}}^+$ does not contain a regular Reeb vector field.*

Proof. Assume that $\xi_{\mathbf{v}} \in \mathfrak{t}_{\mathbf{w}}^+$ is a regular Reeb vector field. Such a $\xi_{\mathbf{v}}$ is regular if and only if $\mathbf{v} = (1, 1)$ and $m = 1$. But this implies that $2 < l^\infty = s = \gcd(s, w^0 - w^\infty)$ so s divides $w^0 - w^\infty$. But also since $c_1(\mathcal{D}_{\mathbf{l}, \mathbf{w}}) = 0$ equations (32) hold. So s divides $|\mathbf{w}| = w^0 + w^\infty$. But since $s > 2$ either s or $\frac{s}{2}$ must divide both w^0 and w^∞ , and this gives a contradiction since $\gcd(w^0, w^\infty) = 1$. \square

2.8. A Categorical Approach. For a fixed quasiregular Sasaki manifold (M, \mathcal{S}) we consider the set of CR orbifolds of Sasaki type

$$(34) \quad \{(M_{\mathbf{l}, \mathbf{w}}, \mathcal{D}_{\mathbf{l}, \mathbf{w}}, J_{\mathbf{w}})\}_{(\mathbf{l}, \mathbf{w}) \in (\mathbb{Z}^+)^2 \times (\mathbb{Z}^+)^2}$$

given by the join construction. Each element $(M_{\mathbf{l}, \mathbf{w}}, \mathcal{D}_{\mathbf{l}, \mathbf{w}}, J_{\mathbf{w}})$ consists of a real 2-parameter family of Sasakian structures parameterized by the \mathbf{w} Sasaki cone $\mathfrak{t}_{\mathbf{w}}^+$ which is completely characterized by its dense subset of quasiregular Reeb vector fields $\xi_{\mathbf{m}}$. The set of all such Sasakian structures

$$\{\mathcal{S}_{\mathbf{l}, \mathbf{w}, \mathbf{m}}\}_{(\mathbf{l}, \mathbf{w}, \mathbf{m}) \in (\mathbb{Z}^+)^2 \times (\mathbb{Z}^+)^2 \times (\mathbb{Z}^+)^2}$$

are the object set $\mathcal{G}_{\mathbf{l}, \mathbf{w}, \mathbf{m}}$ of a groupoid whose morphisms are equivalences, that is, orbifold diffeomorphisms $f : M_{\mathbf{l}, \mathbf{w}} \rightarrow M_{\mathbf{l}', \mathbf{w}'}$ such that

$$(35) \quad f_* \mathcal{D}_{\mathbf{l}, \mathbf{w}} = \mathcal{D}_{\mathbf{l}', \mathbf{w}'}, \quad f_* \circ J_{\mathbf{w}} = J_{\mathbf{w}'} \circ f_*$$

and $f^* \mathcal{S}_{\mathbf{l}', \mathbf{w}', \mathbf{m}'} = \mathcal{S}_{\mathbf{l}, \mathbf{w}, \mathbf{m}}$. Note that the isotropy subgroup of \mathcal{G}_1 at $\mathcal{S}_{\mathbf{l}, \mathbf{w}, \mathbf{m}}$ is just the Sasaki automorphism group $\mathfrak{Aut}(\mathcal{S}_{\mathbf{l}, \mathbf{w}, \mathbf{m}})$. We work with the full subgroupoid $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ whose object set \mathcal{G}_0 satisfies the condition that the components of \mathbf{l} and \mathbf{w} are relatively prime.

Similarly, consider the set of projective algebraic ruled Kähler orbifolds of the form

$$(36) \quad \{\mathcal{Z}_{n, \mathbf{m}} = (S_n, \Delta_{\mathbf{m}} + \pi^{-1}(\Delta_N), \omega_{n, \mathbf{m}, N})\}_{(n, \mathbf{m}) \in \mathbb{Z} \times (\mathbb{Z}^+)^2}$$

that satisfy $[\omega_{n, \mathbf{m}, N}] \in H_{orb}^2(\mathcal{Z}_{n, \mathbf{m}}, \mathbb{Z})$ and $\gcd(m^0, m^\infty, |n|) = 1$ where N is fixed. These form the object set \mathcal{Q}_0 of a groupoid $\mathcal{Q} = \{\mathcal{Q}_{n, \mathbf{m}}\}$ whose morphisms are orbifold biholomorphisms intertwining the Kähler structures.

Proposition 2.21. *There is a full functor $F : \mathcal{G} \rightarrow \mathcal{Q}$.*

Proof. We construct such a functor $F : \mathcal{G} \rightarrow \mathcal{Q}$. Theorems 2.7 and 2.13 shows that each object $\mathcal{S}_{\mathbf{l}, \mathbf{w}, \mathbf{m}}$ of \mathcal{G} uniquely describes a projective algebraic ruled Kähler orbifold of the form $(S_n, \Delta_{\mathbf{m}} + \pi^{-1}(\Delta_N), \omega_{n, \mathbf{m}, N})$ which satisfy the relations of Theorem 2.7. Moreover, morphisms of \mathcal{G} satisfy Equations (35), so they induce morphisms of \mathcal{Q} . Since morphisms of \mathcal{Q} are orbifold biholomorphisms intertwining the Kähler structures, they also induce morphisms of \mathcal{G} . Then Proposition 4.22 of [BHLTF18] shows that this is invertible on the objects in the image of F . That is, F induces a surjective functor

$$\mathrm{Hom}_{\mathcal{G}}(A, B) \rightarrow \mathrm{Hom}_{\mathcal{Q}}(F(A), F(B))$$

so it is full. \square

Remark 2.22. We could also consider groupoids which include equivalences at the level of the orbibundle $M \rightarrow N$, but we do not do so here.

Remark 2.23. Note that different orbifold structures on the same S_n do not necessarily correspond to the same join $M_{\mathbf{l}, \mathbf{w}}$. They can correspond to spaces that are not even homotopy equivalent.

Example 2.2. We give an example of the equivalence of non-trivial equivalences. The involution $(z_1, z_2) \mapsto (z_2, z_1)$ on S^3 induces a diffeomorphism $\iota_{\mathcal{G}}^\perp : M_{1, \mathbf{w}} \rightarrow M_{1, \mathbf{w}^\perp}$ sending $\mathcal{S}_{1, \mathbf{w}, \mathbf{m}}$ to $\mathcal{S}_{1, \mathbf{w}^\perp, \mathbf{m}^\perp}$ where $\mathbf{w}^\perp = (w^\infty, w^0)$ and $\mathbf{m}^\perp = (m^\infty, m^0)$. This gives a nontrivial involution $\iota_{\mathcal{G}}^\perp$ in \mathcal{G}_1 . Similarly, we have an involution $\iota_{\mathcal{Q}}^\perp \in \mathcal{Q}_1$ given by the fiber inversion map which interchanges the branch divisors (D^0, m^0) and (D^∞, m^∞) and sends $(S_n, \Delta_{\mathbf{m}} + \pi^{-1}(\Delta_N), \omega_{n, \mathbf{m}, N})$ to $(S_{-n}, \Delta_{\mathbf{m}^\perp} + \pi^{-1}(\Delta_N), \omega_{-n, \mathbf{m}^\perp, N})$. Using (35) and the relations of Theorem 2.7 we see that $F(\iota_{\mathcal{G}}^\perp) = \iota_{\mathcal{Q}}^\perp$.

To each element $\mathcal{S}_{1, \mathbf{w}, \mathbf{m}} \in \mathcal{G}_0$ we can associate the first Chern class $c_1(\mathcal{D}_{1, \mathbf{w}})$ of the contact bundle which is an invariant under morphisms \mathcal{G}_1 . Similarly, to each element $\mathcal{Z}_{n, \mathbf{m}} \in \mathcal{Q}_0$ we associate $c_1^{orb}(S_n, \Delta_{\mathbf{m}, N})$ which also is invariant under the morphisms \mathcal{Q}_1 . Furthermore, the functor F induces a map

$$(37) \quad F^* c_1^{orb}(S_n, \Delta_{\mathbf{m}, N}) = c_1(\mathcal{D}_{1, \mathbf{w}}).$$

Example 2.3. The $Y^{p,q}$ s and Hirzebruch orbifolds. Here we take $M = S^3$ with its standard SE metric. As shown in Example 6.8 of [BTF16] this corresponds to the join $M \star_1 S_{\mathbf{w}}^3$ with

$$(38) \quad \mathbf{l} = (\gcd(p+q, p-q), p), \quad \mathbf{w} = \frac{1}{\gcd(p+q, p-q)}(p+q, p-q).$$

Here we take $p > 0$ and $-p < q < p$ where $q = 0$ corresponds to $\mathbf{w} = (1, 1)$. The condition $\gcd(l^0, l^\infty) = 1$ corresponds to $\gcd(p, |q|) = 1$ when $q \neq 0$ and $p = 1$ if $q = 0$. Choose a quasiregular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}_{\mathbf{w}}^+$. Then Theorem 2.7 gives $F(Y^{p,q}, \xi_{\mathbf{v}}) = (S_n, \Delta_{\mathbf{m}})$ where S_n is the ruled surface over $\mathbb{C}\mathbb{P}^1$ with orbifold fibers $\mathbb{C}\mathbb{P}^1[\mathbf{w}]/\mathbb{Z}_m$ satisfying $\mathbf{m} = m\mathbf{v}$ and

$$n = l^0 \frac{p(m^\infty - m^0) + q(m^0 + m^\infty)}{p}, \quad \gcd(m^0, m^\infty, n) = 1.$$

Note that n is an integer since $p = ms$ and s divides $(p+q)v^\infty - (p-q)v^0$. By Proposition 2.21 this gives a left invertible functor from the groupoid \mathcal{G} whose objects are $(Y^{p,q}, \xi_{\mathbf{m}})$ to the groupoid \mathcal{Q} whose objects are Hirzebruch orbifolds $(S_n, \Delta_{\mathbf{m}})$ with Kähler form $c_1^{orb}(S_n, \Delta_{\mathbf{m}})$. The equivalence $\iota_{\mathcal{G}}^\perp$ on \mathcal{G} sending the triple (p, q, \mathbf{m}) to $(p, -q, \mathbf{m}^\perp)$ maps to the equivalence $\iota_{\mathcal{Q}}^\perp$ on \mathcal{Q} sending (n, \mathbf{m}) to $(-n, \mathbf{m}^\perp)$.

3. EXTREMAL, CSC, AND KE ADMISSIBLE METRICS ON THE $S_{\mathbf{w}}^3$ -JOIN

An important property of the $S_{\mathbf{w}}^3$ join construction is that one can apply the admissible constructions from Section 2.6 to give explicit constructions of existence theorems for extremal and constant scalar curvature Sasaki metrics. This was presented in a recent survey [BTF15] as well as our original papers [BTF14a, BTF16] when M is regular; however, as discussed above in Theorem 2.7 the quasi-regular case goes through without much change.

In particular, the lifted admissible quasi-regular structure in Theorem 2.13

- has a lifted extremal polynomial $\tilde{F}_{ext}(\mathfrak{z}) = mv^0 v^\infty p_{\mathbf{v}}^*(F_{ext}(\mathfrak{z}))$ whose positivity (or not) for $-1 < \mathfrak{z} < 1$ will determine whether the corresponding expression (27) gives a genuine quasi-regular admissible extremal Sasaki metric within the isotopy class.

- Setting $b = v^\infty/v^0$, (27) with $\tilde{\Theta}(\mathfrak{z}) = \frac{\tilde{F}_{ext}(\mathfrak{z})}{(1+r\mathfrak{z})^{d_N}}$ gives a genuine CSC Sasaki metric if and only if b is a root of f where

$$\begin{aligned}
(39) \quad f(b) &= (w^0)^{2(d_N+1)} b^{2d_N+3} (A_N l^\infty + l^0 (d_N + 1) w^\infty - b (d_N + 1) l^0 w^0) \\
&- (w^0)^{d_N+2} (w^\infty)^{d_N} b^{d_N+3} (d_N + 1) (A_N (d_N + 1) l^\infty - l^0 ((d_N + 1) w^0 + (d_N + 2) w^\infty)) \\
&+ (w^0)^{d_N+1} (w^\infty)^{d_N+1} b^{d_N+2} (2A_N d_N (d_N + 2) l^\infty - (d_N + 1) (2d_N + 3) l^0 (w^0 + w^\infty)) \\
&- (w^0)^{d_N} (w^\infty)^{d_N+2} b^{d_N+1} (d_N + 1) (A_N (d_N + 1) l^\infty - l^0 ((d_N + 2) w^0 + (d_N + 1) w^\infty)) \\
&+ (w^\infty)^{2(d_N+1)} (b (A_N l^\infty + l^0 (d_N + 1) w^0) - (d_N + 1) l^0 w^\infty).
\end{aligned}$$

- Assume further that M is an S^1 orbifold over a compact positive Kähler-Einstein orbifold N with Kähler class $[\omega_N] \in H^2_{orb}(N, \mathbb{Z})$ and that the relatively prime positive integers (l^0, l^∞) are the relative Fano indices given explicitly by

$$l^0 = \frac{\mathcal{J}_N}{\gcd(w^0 + w^\infty, \mathcal{J}_N)}, \quad l^\infty = \frac{w^0 + w^\infty}{\gcd(w^0 + w^\infty, \mathcal{J}_N)},$$

where \mathcal{J}_N denotes the orbifold Fano index of N . Then (27) with $\tilde{\Theta}(\mathfrak{z}) = \frac{\tilde{F}_{ext}(\mathfrak{z})}{(1+r\mathfrak{z})^{d_N}}$ is an η -Einstein Sasaki metric if and only if

$$(40) \quad \int_{-1}^1 ((1-b) - (1+b)\mathfrak{z}) ((b+t) + (b-t)\mathfrak{z})^{d_N} d\mathfrak{z} = 0,$$

where $t = w^\infty/w^0$ is assumed to satisfy $0 < t < 1$ and (as above) $b = v^\infty/v^0$. Naturally, in this case, b is also a root of f in (39).

Remark 3.1. The analysis on pages 1053-1054 of [BTF16] shows that in the orbifold category there are infinitely many quasi-regular Sasaki- η -Einstein manifolds of the form $M \star_{\mathbf{1}} S^3_{\mathbf{w}}$. The condition that the join $M \star_{\mathbf{1}} S^3_{\mathbf{w}}$ be smooth is given by (9), and the condition for a Sasaki-Einstein structure is that the components l^0, l^∞ of \mathbf{l} are the relative Fano indices given in Theorem 3.3. So such an $M_{\mathbf{1}, \mathbf{w}}$ with quasiregular Sasakian structure \mathcal{S} is smooth if and only if

$$(41) \quad \gcd(\Upsilon_{\mathcal{S}}, l^0 w^0 w^\infty) = 1$$

where $\Upsilon_{\mathcal{S}} = \Upsilon_N$ denotes the order of the Sasaki manifold (M, \mathcal{S}) .

In Section 4 we will explore some remarkable polynomials arising from (40).

3.1. The Apostolov-Calderbank CR -twist and irregular solutions. In a recent paper of V. Apostolov and D. M. J. Calderbank [AC21] the notion of a CR -twist was introduced. See Definition 4 in [AC21]. This casts an illuminating light on the \mathbf{w} -cone for the join $M \star_{\mathbf{1}} S^3_{\mathbf{w}}$. Indeed, following Section 4.5 of [AC21]³ and bearing in mind equation (26) we see in broad strokes the following:

³We would also like to thank Vestislav Apostolov for a very helpful e-mail conversation.

For a fixed CR -structure, the orbifold-Kähler quotient of each quasi-regular ray given by a pair of co-prime (v^0, v^∞) is actually a CR -twist by a positive Killing potential f of the product Kähler quotient of the join with respect to the original ray (given by (w^0, w^∞)). Up to homothety, $f = \tilde{\mathfrak{z}} + 1/r$, where we now view $\tilde{\mathfrak{z}}$ as the lift of the moment map of $(\mathbb{C}\mathbb{P}^1[\mathbf{w}], \omega_{\mathbf{w}})$ to the product. As we know from Section 2.5,

$$1/r = \frac{w^0 v^\infty + w^\infty v^0}{w^0 v^\infty - w^\infty v^0} = \frac{w^0 b + w^\infty}{w^0 b - w^\infty},$$

where $b = v^\infty/v^0$ as above. Allowing b to be irrational, corresponds to letting $1/r$ be irrational and thus the - locally defined - Kähler quotients of irregular rays corresponding to $b \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ are also captured as CR -twists by $f = \tilde{\mathfrak{z}} + 1/r$.

Following [AC21], the construction of the quasi-regular admissible extremal Sasaki metrics from above may now be seen as arising as follows. By Theorem 1 in [AC21] and the fact that all the Reeb vector fields in the \mathbf{w} -cone commute with $\xi_{\mathbf{w}}$, the Reeb vector field $\xi_{\mathbf{v}}$ determined by $f = \tilde{\mathfrak{z}} + 1/r$ is extremal if and only if $\xi_{\mathbf{w}}$ is so-called $(\xi_{\mathbf{v}}, d_N + 3)$ -extremal (which in turn means that the quotient product metric is $(f, d_N + 3)$ -extremal). The latter condition does not hold if we use the canonical Sasaki Extremal structure on $S_{\mathbf{w}}^3$, but one can change the profile function from the one of the original extremal Kähler structure $(\omega_{\mathbf{w}}, g_{\mathbf{w}})$ to one (\tilde{A} from (17) in [AC21]) that does satisfy that the resulting product metric on $N \times \mathbb{C}\mathbb{P}^1[\mathbf{w}]$ is $(f, d_N + 3)$ -extremal. The only “catch”, is that this new profile function may not be positive everywhere. Ignoring this worry for a moment, we now have changed the CR -structure of the original join but we remain in the same isotopy class (with the usual “symplectic to complex” viewpoint shift) and the resulting Sasaki structure at $\xi_{\mathbf{v}}$ looks like (27) with $\tilde{\Theta}(\tilde{\mathfrak{z}}) = \frac{\tilde{F}_{ext}(\tilde{\mathfrak{z}})}{(1+r\tilde{\mathfrak{z}})^{d_N}}$. Now the positivity condition of the aforementioned profile function (the \tilde{A}) is equivalent to the positivity condition of $\tilde{F}_{ext}(\tilde{\mathfrak{z}})$.

One great advantage of keeping the change of the CR -structure at the $\xi_{\mathbf{w}}$ -ray site and then “do a twist” to $\xi_{\mathbf{v}}$, is that it becomes very clear why we may extend our constructions of extremal admissible Sasaki metrics to $b \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ (as we did in [BTF16]). It simply corresponds to allowing the number $\tilde{b}_0 := 1/r$ to be in $\mathbb{R}^+ \setminus \mathbb{Q}^+$ and constructing $(\tilde{\mathfrak{z}} + \tilde{b}_0, d_N + 3)$ -extremal quasi-regular Kähler metrics on the product $N \times \mathbb{C}\mathbb{P}^1[\mathbf{w}]$. The profile function \tilde{A} from (17) of [AC21], clearly varies smoothly with \tilde{b}_0 and, in turn, so the resulting admissible extremal metric at $\xi_{\mathbf{v}}$, with corresponding $\tilde{\Theta}(\tilde{\mathfrak{z}}) = \frac{\tilde{F}_{ext}(\tilde{\mathfrak{z}})}{(1+r\tilde{\mathfrak{z}})^{d_N}}$, will vary smoothly with (v^0, v^∞) . In particular, when $b = v^\infty/v^0$ and b is a root of f in (39) then for $b \in \mathbb{Q}^+$ we have a quasi-regular Sasaki CSC metric, whereas when $b \in \mathbb{R}^+ \setminus \mathbb{Q}^+$, we have an irregular Sasaki CSC metric. Likewise, if b is a root of f in (40), and the other conditions in the bullet point containing (40) are satisfied, then for $b \in \mathbb{Q}^+$ we have a quasi-regular η -Einstein Sasaki metric, whereas when $b \in \mathbb{R}^+ \setminus \mathbb{Q}^+$, we have an irregular η -Einstein Sasaki metric.

3.2. Constant Scalar Curvature. Summarizing our results we see that by following the same arguments as in Section 6.2 of [BTF16] we have orbifold versions of Theorems 1.1 through 1.4. of [BTF16]:

Theorem 3.2. *Let $M_{1,\mathbf{w}} = M \star_1 S_{\mathbf{w}}^3$ be the $S_{\mathbf{w}}^3$ -join with a quasi-regular Sasaki manifold (orbifold) M which is an S^1 orbibundle over a compact Kähler orbifold N with constant*

scalar curvature s_N . Then for each vector $\mathbf{w} = (w^0, w^\infty) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w^0 > w^\infty$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in the 2-dimensional \mathbf{w} -Sasaki cone on $M_{\mathbf{1},\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has constant scalar curvature. Moreover, if $s_N \geq 0$, then the \mathbf{w} -Sasaki cone $\mathfrak{t}_{\mathbf{w}}^+$ is exhausted by extremal Sasaki metrics. If $s_N > 0$ and l^∞ is sufficiently large then the \mathbf{w} -cone has at least 3 cscS rays.

Theorem 3.3. *Let $M_{\mathbf{1},\mathbf{w}} = M \star_{\mathbf{1}} S_{\mathbf{w}}^3$ be the $S_{\mathbf{w}}^3$ -join with a quasi-regular Sasaki manifold (orbifold) M which is an S^1 orbibundle over a compact positive Kähler-Einstein orbifold N with primitive Kähler class $[\omega_N] \in H_{orb}^2(N, \mathbb{Z})$. Assume that the relatively prime positive integers (l^0, l^∞) are the relative Fano indices given explicitly by*

$$l^0 = \frac{J_N}{\gcd(w^0 + w^\infty, J_N)}, \quad l^\infty = \frac{w^0 + w^\infty}{\gcd(w^0 + w^\infty, J_N)},$$

where J_N denotes the orbifold Fano index of N . Then for each vector $\mathbf{w} = (w^0, w^\infty) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w^0 > w^\infty$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in the 2-dimensional \mathbf{w} -Sasaki cone $\mathfrak{t}_{\mathbf{w}}^+$ on $M_{\mathbf{1},\mathbf{w}}$ such that the corresponding Sasakian structure $\mathcal{S}_{\mathbf{v}} = (\xi_{\mathbf{v}}, \eta_{\mathbf{v}}, \Phi, g)$ is Sasaki-Einstein. Moreover, this ray is the only admissible CSC ray in the Sasaki cone.

Remark 3.4. The constant scalar curvature ray in Theorems 3.2 and 3.3 can be either quasi-regular or irregular. In the former case the procedure can be iterated.

In dimension 5 we have

Corollary 3.5. *A cone decomposable compact Sasaki 5-manifold M^5 is diffeomorphic to a lens space bundle over a Riemann surface of genus g . Moreover, if $g > 0$ it admits a cscS ray in its Sasaki cone; whereas, if $g = 0$ choosing the unique ray of Sasakian structures with constant Φ sectional curvature on M^3 (where $M^5 = M^3 \star_{\mathbf{1}} S_{\mathbf{w}}^3$) gives a constant scalar curvature Sasaki metric on M^5 .*

Proof. By definition M^5 is the join $M^3 \star_{\mathbf{1}} S_{\mathbf{w}}^3$ where M^3 is a Sasaki 3-manifold. By [Gei97] M^3 is diffeomorphic to one of the three 3-manifolds

$$\Gamma \backslash S^3, \quad \Gamma \backslash \mathcal{H}^3, \quad \Gamma \backslash \widetilde{SL}(2, \mathbb{R})$$

where $\widetilde{SL}(2, \mathbb{R})$ is the universal cover of $SL(2, \mathbb{R})$, \mathcal{H}^3 is the Heisenberg group, and Γ is a discrete group of isometries with respect to the Sasaki metric. These are all S^1 Seifert bundles over a Riemann surface of genus g where $g = 0$ in the first case, $g = 1$ in the second, and $g > 1$ in the third. By [Bel01] for cases 2 and 3 there is a contact isotopy to a Sasaki metric of constant Φ sectional curvature. Whereas, if $g = 0$ there is a unique ray of constant Φ sectional curvature in the Sasaki cone. The corollary then follows from Theorem 3.2 and [BTF14a]. \square

Remark 3.6. The analogue of this corollary is definitely false in higher dimensions.

Remark 3.7. It should be noted that the methods of [AC21] also has the advantage of being able to go beyond the \mathbf{w} -cone (within the full Sasaki-cone), for the join $M^3 \star_{\mathbf{1}} S_{\mathbf{w}}^3$, whenever the set of Killing potentials of (N, ω_N, γ_N) is non-empty. This could be useful in, for example, exploring illuminating examples for the open questions in [BHLTF21].

4. THE S- η -E POLYNOMIALS

We now consider the S- η -E condition and equation (40) more closely. We choose a Sasakian structure $\mathcal{S}_{\mathbf{v}}$ in the \mathbf{w} -cone of $M_{1,\mathbf{w}}$ with Reeb field $\xi_{\mathbf{v}}$. Following the analysis on pages 1053-1054 [BTF16] we define $b = \frac{v^\infty}{v^0} = kt_{\mathbf{w}}$ and $t_{\mathbf{w}} = \frac{w^\infty}{w^0}$ together with the polynomials of degree $d := d_N$

$$(42) \quad p^\pm(k) = \int_{-1}^1 (1 \pm \mathfrak{z})((k+1) + (k-1)\mathfrak{z})^d d\mathfrak{z}.$$

Then equation (40) is equivalent with $b = \frac{p^-(k)}{p^+(k)}$ and $t_{\mathbf{w}} = \frac{p^-(k)}{kp^+(k)}$. We can easily compute the polynomials $p^\pm(k)$ explicitly.

Lemma 4.1. *The polynomials $p^\pm(k)$ defined in (42) satisfy*

$$p^-(k) = \frac{2^{d+2}}{(d+1)(d+2)} \sum_{j=0}^d (d+1-j)k^j, \quad p^+(k) = \frac{2^{d+2}}{(d+1)(d+2)} \sum_{j=0}^d (j+1)k^j,$$

and

$$(43) \quad \frac{p^-(k)}{p^+(k)} = \frac{d+1+dk+\dots+2k^{d-1}+k^d}{1+2k+\dots+(d+1)k^d}.$$

Proof. Performing the integration in (42) explicitly we find

$$\begin{aligned} p^+(k) &= \frac{2^{d+2}}{(d+1)(d+2)} \frac{(d+1)k^{d+2} - (d+2)k^{d+1} + 1}{(k-1)^2}, \\ p^-(k) &= \frac{2^{d+2}}{(d+1)(d+2)} \frac{k^{d+2} - (d+2)k + d+1}{(k-1)^2}, \end{aligned}$$

from which the expressions for $p^\pm(k)$ are easily verified. \square

We can use transverse scaling to reduce our discussion to the 1 dimensional space of rays $\mathfrak{R}_{\mathbf{w}}$ of $\mathfrak{t}_{\mathbf{w}}^+$ which is uniquely determined by its slope. We can thus ignore the factor $\frac{2^d}{(d+1)(d+2)}$ in front of the sums in $p^\pm(k)$.

Then as in [BTF16] we have

Theorem 4.2. *Consider Sasaki join manifolds $M_{1,\mathbf{w}}$ with $w^0 > w^\infty$ relatively prime positive integers and assume $c_1(\mathcal{D}_{1,\mathbf{w}}) = 0$. Then for each pair (w^0, w^∞) of relatively prime positive integers with $w^0 > w^\infty$ there exists a unique S- η -E ray $\mathfrak{r}_{\mathbf{v}}$ where $\mathbf{v} = (v^0, v^\infty) = (p^+(k), p^-(k))$ which corresponds to the unique root $k \in (1, \infty)$ of the polynomial*

$$(44) \quad \mathcal{P}_{\mathbf{w}}(k) = w^\infty(d+1)k^{d+1} + \sum_{j=0}^d (|\mathbf{w}|j - w^0(d+1))k^j.$$

Furthermore, the S- η -E ray is quasi-regular if and only if $k \in (1, \infty) \cap \mathbb{Q}$.

Proof. Given the relatively prime integers $w^0 > w^\infty$ it follows as in Section 6.2 of [BTF16] that there is a unique S- η -E ray corresponding to $k \in (1, \infty)$ such that

$$(45) \quad w^\infty kp^+(k) = w^0 p^-(k).$$

It follows easily from Lemma 4.1 that the solutions of (45) correspond precisely to roots of the polynomial $\mathcal{P}_{\mathbf{w}}$. This correspondence is that the ray $\mathfrak{r}_{\mathbf{v}}$ with $\mathbf{v} = (p^+(k), p^-(k))$ corresponds to the root $k \in (1, \infty)$. That there exists a root $k \in (1, \infty)$ follows since

$$\mathcal{P}_{\mathbf{w}}(1) = -\frac{(d+1)(d+2)}{2}(w^0 - w^\infty) < 0$$

and $\lim_{k \rightarrow \infty} \mathcal{P}_{\mathbf{w}}(k) = +\infty$. Moreover, this root is unique since roots of $\mathcal{P}_{\mathbf{w}}$ in $(1, \infty)$ correspond to S- η -E rays which by [MSY08] are unique.

Since $\mathbf{v} = (v^0, v^\infty)$ parameterizes the \mathbf{w} cone $\mathfrak{t}_{\mathbf{w}}^+$ on $M_{1, \mathbf{w}}$, we see that a Reeb vector field $\xi_{\mathbf{v}}$ is quasi-regular if and only if \mathbf{v} lies on the integral lattice $(\mathbb{Z}^+)^2 \subset (\mathbb{R}^+)^2$. Otherwise $\xi_{\mathbf{v}}$ will be irregular. Since we use the convention $w^0 > w^\infty$ this translates to: $\xi_{\mathbf{v}}$ is quasiregular if and only if $k \in (1, \infty) \cap \mathbb{Q}$. \square

Remark 4.3. From the discussion in Section 3.1 in [BTF16] it follows that the Gorenstein condition $c_1(\mathcal{D}_{1, \mathbf{w}}) = 0$ is equivalent to the conditions at the beginning of Theorem 3.3 and thus Theorem 4.2 should be seen as a more explicit exposition of the Sasaki η -Einstein solutions implied by Theorem 3.3. It is straightforward to generalize these results to the \mathbb{Q} -Gorenstein case when $c_1(\mathcal{D}_{1, \mathbf{w}})$ is a torsion class.

Remark 4.4. The unique almost regular ray $((v^0, v^\infty) = (1, 1)$, i.e., $b = 1$) is never S- η -E (since $\mathcal{P}_{\mathbf{w}}(1) \neq 0$).

Remark 4.5. Note that if $k \in \mathbb{Z}^+$, \mathbf{w} satisfies

$$(46) \quad \mathbf{w} = (w^0, w^\infty) = \left(\frac{kp^+(k)}{(\gcd(p^-(k), kp^+(k)))}, \frac{p^-(k)}{(\gcd(p^-(k), kp^+(k)))} \right).$$

We can give a slightly different way of treating quasi-regular S- η -E rays. Note that if $k = \frac{p}{q}$ is any rational number and we define

$$F(p, q) = (d+1)q^d + dq^{d-1}p + \cdots + 2qp^{d-1} + p^d$$

then we have

$$(47) \quad \frac{p^-(k)}{p^+(k)} = \frac{F(p, q)}{F(q, p)}.$$

We define the map $\kappa : (1, \infty) \cap \mathbb{Q} \rightarrow \mathfrak{R}_{\mathbf{w}} \approx \mathbb{R}^+$ by

$$(48) \quad \kappa(k) = \frac{d+1+dk+\cdots+2k^{d-1}+k^d}{1+2k+\cdots+dk^{d-1}+(d+1)k^d} = \frac{v^\infty}{v^0}.$$

We can identify $(1, \infty) \cap \mathbb{Q}$ with the set \mathcal{R} of pairs (p, q) of relatively prime positive integers with $p > q$. We have an injective map $\kappa : \mathcal{R} \rightarrow \mathcal{R}$ defined by

$$(49) \quad \kappa(p, q) = \left(\frac{F(q, p)}{C}, \frac{F(p, q)}{C} \right) = (v^0, v^\infty)$$

where $C = \gcd(F(q, p), F(p, q))$. Then we have

Corollary 4.6. *A quasi-regular ray $\mathfrak{r}_{\mathbf{v}}$ defined by $\mathbf{v} \in \mathcal{R}$ is S- η -E if and only if $\mathbf{v} \in \text{image of } \kappa \text{ and satisfies the constraint}$*

$$(50) \quad \frac{w^\infty}{w^0} = \frac{qF(p, q)}{pF(q, p)} = \frac{qv^\infty}{pv^0}.$$

5. RELATION WITH K-STABILITY

The purpose of this section is to show that in a well defined sense the $S_{\mathbf{w}}^3$ join operation preserves certain K-stability properties. For quasi-regular Sasaki metrics such stability properties are equivalent to the stability properties of certain Kähler orbifolds which was investigated in detail in [RT11]. However, to treat the general Sasaki metrics one needs to work on the affine cone. For any contact manifold M we consider the cone $C(M) = M \times \mathbb{R}^+$. Choosing a 1-form η in the contact structure of M we form an exact symplectic structure on $C(M)$ by defining $\omega = d(r^2\eta)$ where $r \in \mathbb{R}^+$. The pair $(C(M), \omega)$ is called the symplectization of the (strict) contact structure (M, η) . Using the Liouville vector field $\Psi = r\partial_r$, we define a natural almost complex structure I on $C(M)$ by

$$IX = \Phi X + \eta(X)\Psi, \quad I\Psi = -\xi$$

where X is a vector field on M , and the Reeb field ξ is understood to be lifted to $C(M)$. Without further ado we shall identify M with $M \times \{r = 1\} \subset C(M)$. By adding the cone point we obtain an affine variety $Y = C(M) \cup \{0\}$ which is invariant under the complexification $\mathbb{T}^{\mathbb{C}}$ of the torus action \mathbb{T}^k . This was done by Collins and Székelyhidi [CS18] and it is this approach that we follow here. We begin by defining two important functionals, the *total volume* and the *total transverse scalar curvature*, viz.

$$(51) \quad \mathbf{V}_{\xi} = \int_M dv_g, \quad \mathbf{S}_{\xi} = \int_M s^T dv_g$$

where s^T denotes the transverse scalar curvature of the Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$. We define the *Donaldson-Futaki invariant* on the polarized affine cone (Y, ξ) by

$$(52) \quad \text{Fut}(Y, \xi, a) := \frac{\mathbf{V}_{\xi}}{n} D_a \left(\frac{\mathbf{S}_{\xi}}{\mathbf{V}_{\xi}} \right) + \frac{\mathbf{S}_{\xi} D_a \mathbf{V}_{\xi}}{n(n+1)\mathbf{V}_{\xi}}.$$

To proceed further we need the definition of a special type of degeneration known as a *test configuration* due to Donaldson [Don02] in the Kähler case and Collins-Székelyhidi [CS18] in the Sasaki case:

Definition 5.1. Let (Y, ξ) be a polarized affine variety with an action of a torus $T^{\mathbb{C}^*}$ for which the Lie algebra \mathfrak{t} of a maximal compact subtorus $T^{\mathbb{R}}$ contains the Reeb vector field ξ . A $T^{\mathbb{C}}$ -equivariant test configuration for (Y, ξ) is given by a set of k $T^{\mathbb{C}}$ -invariant generators f_1, \dots, f_k of the coordinate ring \mathcal{H} of Y and k integers w_1, \dots, w_k (weights). The functions f_1, \dots, f_k are used to embed Y in \mathbb{C}^k on which the weights w_1, \dots, w_k determine a \mathbb{C}^* action. By taking the flat limit of the orbits of Y to $0 \in \mathbb{C}$ we get a family of affine schemes $\mathcal{Y} \rightarrow \mathbb{C}$. There is then an action of \mathbb{C}^* on the ‘central fiber’ Y_0 , generated by $a \in \mathfrak{t}$, the Lie algebra of some torus $T^{\mathbb{C}} \subset \text{GL}(k, \mathbb{C})$ containing $T^{\mathbb{C}}$.

We now obtain the correct notions of K-stability following [Tia97, Don02, CS18, CS19]

Definition 5.2. We say that the polarized affine variety (Y, ξ) is *K-semistable* if for each $T^{\mathbb{C}}$ such that $\xi \in \mathfrak{t}$ the Lie algebra of $T^{\mathbb{R}}$ and any $T^{\mathbb{C}}$ -equivariant test configuration we have

$$(53) \quad \text{Fut}(Y, \xi, a) \geq 0$$

where $a \in \mathfrak{t}$ is the infinitesimal generator of the induced S^1 action on the central fiber Y_0 . The polarized variety (Y, ξ) is said to be *K-polystable* if equality holds only for the product configuration $\mathcal{Y} = Y \times \mathbb{C}$.

The general Yau-Tian-Donaldson conjecture in the Sasaki case, which says that cscS metrics corresponds to some kind of K-stability on the affine cone, is still open. It has been affirmed in one direction by Collins and Székelyhidi.

Theorem 5.3 ([CS18]). *Let (M, \mathcal{S}) be a Sasaki manifold of constant scalar curvature. Then its polarized affine cone (Y, ξ) is K-semistable.*

Since there is a one to one correspondence between a Sasaki manifold and its polarized affine variety, we can also refer to this as the K-semistability of the corresponding Sasaki manifold. We now have an immediate corollary of Theorems 3.2 and 5.3:

Corollary 5.4. *Let M be a quasi-regular Sasaki manifold with constant scalar curvature, let $Y_{1, \mathbf{w}}$ denote the affine cone of the $S_{\mathbf{w}}^3$ join $M_{1, \mathbf{w}} = M \star_1 S_{\mathbf{w}}^3$. Then there exists a K-semistable polarization ξ of $Y_{1, \mathbf{w}}$, or equivalently a K-semistable Sasakian structure (ξ, η, Φ, g) on $M_{1, \mathbf{w}}$.*

In the \mathbb{Q} -Gorenstein case, i.e. $c_1(Y)$ is a torsion class, it is well known that the existence of Sasaki-Einstein metrics of M is equivalent to the existence of Ricci flat Kähler metrics on the affine cone $C(M) = M \times \mathbb{R}^+$, and it is convenient to include the “cone point” and define $Y = C(M) \cup \{0\}$. As before the pair (Y, ξ) is a polarized affine variety, and we can also assume that ξ belongs to the Lie algebra \mathfrak{t} of a maximal torus \mathbb{T} . However, in this case we have a nowhere vanishing section Ω of the canonical line bundle $\Lambda^{n+1, 0}(C(M))$ that satisfies

$$\mathcal{L}_\xi \Omega = i(n+1)m\Omega$$

for some $m \in \mathbb{Z}^+$. Collins and Székelyhidi call this pair (Y, ξ) a *normalized Fano cone singularity*, and they restrict their test configurations to those with normal central fibers.

Definition 5.5. A normalized Fano cone singularity (Y, ξ) is called a \mathbb{T} -equivariantly *K-stable Reeb polarization* if for all test configurations with normal central fiber (Y_0, ξ) equation (53) holds with equality only if (Y, ξ) is isomorphic to (Y_0, ξ) .

This gives the Sasaki version of the famous Chen-Donaldson-Sun theorem.

Theorem 5.6 ([CS19]). *Let (M, \mathcal{S}) be a Sasaki manifold with $c_1(\mathcal{D})$ a torsion class. Then (M, \mathcal{S}) admits a Sasaki-Einstein metric if and only if its polarized affine cone (Y, ξ) is \mathbb{T} -equivariantly K-stable.*

Combining this theorem with Theorem 3.3 gives

Corollary 5.7. *The $S_{\mathbf{w}}^3$ -join, $M_{1, \mathbf{w}} = M \star_1 S_{\mathbf{w}}^3$, described in Theorem 3.3 admits a \mathbb{T} -equivariantly K-stable Reeb polarization.*

Remark 5.8. By relaxing the primitivity assumption in Theorem 3.3 this corollary easily generalizes to the \mathbb{Q} -Gorenstein case with $c_1(\mathcal{D})$ a torsion class.

Remark 5.9. In both Corollaries 5.4 and 5.7 the K-(semi)stable Sasakian structures on $M_{1, \mathbf{w}}$ could be irregular.

6. APPLICATIONS

It is the purpose of this section to apply our main Theorems 3.2 and 3.3 to give explicit examples of cone decomposable Sasakian structures that admit constant scalar curvature or S- η -E metrics. As shown by Corollary 3.5 cone decomposability is well understood in dimension 5, so we concentrate here on dimension 7 and higher. We shall only give explicit examples in dimension 7, since the complexity increases greatly with dimension.

6.1. Toric Examples. Recall that toric contact structures of Reeb type have families of Sasakian structures [BG00a, Ler02]. It is easy to see that the $S_{\mathbf{w}}^3$ join of a toric contact structure of Reeb type is a toric contact structure of Reeb type. In dimension 5 all toric contact structures of Reeb type on S^3 bundles over S^2 can be realized as joins and have been studied in [Leg11a, BP14]. Examples in dimension 7 can easily be constructed for example $Y^{p,q} \star_1 S_{\mathbf{w}}^3$ as in [BTF19]. This is a special case of the iterated S^3 joins which are treated in [BTF20a].

6.2. Complexity One Examples in Dimension 7. Here we consider the $S_{\mathbf{w}}^3$ join with a complexity one Sasaki 5-manifold recently studied in [CS19]. It is the link $L_{p,q}$ of the Brieskorn varieties

$$(54) \quad f_{p,q} = z_0^p + z_1^q + z_2^2 + z_3^2 = 0$$

of degree $d = 2pq$ with its naturally induced Sasakian structure with Reeb vector field ξ_0 . We denote this join by $M_{p,q,1,\mathbf{w}}$. Topologically the 5-manifolds $L_{p,q}$ are precisely the k -fold connected sums $k(S^2 \times S^3)$ with $k = \gcd(p, q) - 1$ and $k = 0$ means the five sphere S^5 . So from Proposition 2.4 we see that the 7-manifold $M_{p,q,1,\mathbf{w}}$ is simply connected with $\pi_2(M_{p,q,\mathbf{w}}) = H_2(M_{p,q,1,\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}^k$.

The weight vector of the naturally induced Reeb vector field ξ_0 is $(2q, 2p, pq, pq)$. The Sasaki cone $\mathfrak{t}^+(L_{p,q})$ of the link $L_{p,q}$ is 2 dimensional and was computed in [CS19, BvC18] using the Hilbert series of the coordinate ring on the polarized affine variety $(Y_{p,q}, \xi_0) = (\mathbb{C}[z_0, z_1, z_2, z_3]/(f_{p,q}), \xi_0)$. It is given by

$$(55) \quad \mathfrak{t}^+(L_{p,q}) = \left\{ \xi = a\xi_0 + b\zeta \mid a > 0, -\frac{ad}{2} < b < \frac{ad}{2} \right\}$$

where $\zeta \in \mathfrak{t}$ has weight $(1, -1)$ with respect to $(u = z_2 + iz_3, v = z_2 - iz_3)$ and 0 elsewhere. In all cases there is a unique ray that minimizes the volume on the planar slice fixed by a which corresponds to $b = 0$. Moreover, a Theorem of Collins and Székelyhidi [CS19] says that this ray is cscS, hence S- η -E, if and only if $2p > q$ and $2q > p$, or the quadric⁴ $p = q = 2$. The Fano index $J_{p,q}$ of this ray is

$$(56) \quad J_{p,q} = 2q + 2p + pq + pq - d = 2(p + q).$$

Thus, as in Equation (9) of [BTF16] we have

Lemma 6.1. *There is a generator $\gamma \in H^2(M_{p,q,1,\mathbf{w}}, \mathbb{Z})$ such that*

$$c_1(\mathcal{D}_{p,q,1,\mathbf{w}}) = (2l^\infty(p + q) - l^0|\mathbf{w}|)\gamma.$$

Moreover, the second Stiefel-Whitney class is $w_2(M_{p,q,1,\mathbf{w}}) = l^0|\mathbf{w}|\gamma \pmod{2}$.

⁴We exclude the case $p = q = 2$ in what follows. This gives the homogeneous Einstein metric on $S^2 \times S^3$, hence, it is regular and thus treated in [BTF16].

Remark 6.2. For those contact structures $\mathcal{D}_{p,q,\mathbf{l},\mathbf{w}}$ such that $2(p+q)l^\infty > l^0 w^0$ the entire \mathbf{w} Sasaki cone $\mathfrak{t}_{\mathbf{w}}^+$ consists of positive Sasakian structures. When this inequality does not hold type changing occurs. See Proposition 4.2 of [BTF20b].

We can now apply Theorem 3.2 to give a cscS metric in the \mathbf{w} Sasaki cone $\mathfrak{t}_{\mathbf{w}}^+$ of the join $M_{p,q,\mathbf{l},\mathbf{w}}$. But first we construct the S^1 quotient N of $L_{p,q}$ as a log pair.

Lemma 6.3. *Let $k = \gcd(p, q) - 1$, then the quotient orbifold (N, Δ) generated by the Reeb vector field ξ_0 is $(N, \Delta_{p,q})$ where N is the weighted projective hypersurface embedded in $\mathbb{C}\mathbb{P}^3[2, 2, k+1, k+1]$ as $z_0^{k+1} + z_1^{k+1} + z_2^2 + z_3^2 = 0$ and $\Delta_{p,q}$ is the branch divisor*

$$\Delta_{p,q} = \left(1 - \frac{1}{p}\right)(z_0 = 0) + \left(1 - \frac{1}{q}\right)(z_1 = 0).$$

Moreover, there are algebraic quotient singularities with isotropy \mathbb{Z}_{k+1} occurring at the intersection of the irreducible components of $\Delta_{p,q}$, namely the two points $[0, 0, 1, \pm i] \in N$, and \mathbb{Z}_2 quotient singularities at the points at the $k+1$ points $[1, e^{\frac{i\pi(2m+1)}{k+1}}, 0, 0]$, $m = 0, \dots, k$.

Proof. There are three cases to consider, p, q both odd, p, q both even, and p, q have different parity. In the last two cases, the weights have a common factor of 4 and 2, respectively which should be factored out of the weights and the degree. Once this is done the proof uses standard techniques described in Chapter 4 of [BG08], in particular Lemma 5.4.10. First we define a new weight vector $\bar{\mathbf{w}}$ as described in Definition 4.5.7 of [BG08] which by a result of Delorme (Proposition 4.5.10 in [BG08]) gives an isomorphism $\mathbb{C}\mathbb{P}^3[\mathbf{w}] \approx \mathbb{C}\mathbb{P}^3[\bar{\mathbf{w}}]$ as algebraic varieties. For example if p and q are both odd we have from Definition 4.5.7 that

$$\bar{\mathbf{w}} = (2, 2, \gcd(p, q), \gcd(p, q)) = (2, 2, k+1, k+1)$$

and then Lemma 5.4.10 determines the hypersurface. The other two cases are similar once the factorization described above is done. The second statement then follows since the hypersurface inherits the algebraic quotient singularities of $\mathbb{C}\mathbb{P}^3[2, 2, k+1, k+1]$. \square

This gives

Theorem 6.4. *If $2p > q$ and $2q > p$ then for each of the relatively prime pairs (l^0, l^∞) and (w^0, w^∞) there exists a ray of constant scalar curvature Sasakian structures in $\mathfrak{t}_{\mathbf{w}}^+$ on the 7 dimensional Sasaki orbifolds $M_{p,q,\mathbf{l},\mathbf{w}}$, and the entire \mathbf{w} cone $\mathfrak{t}_{\mathbf{w}}^+$ is exhausted by extremal Sasaki metrics. Moreover, if l^∞ is large enough there are at least 3 cscS metrics in $\mathfrak{t}_{\mathbf{w}}^+$. $M_{p,q,\mathbf{l},\mathbf{w}}$ is a smooth manifold if and only if $\gcd(2l^\infty pq, l^0 w^0 w^\infty) = 1$.*

Remark 6.5. It is interesting to note that the last statement of the Theorem together with Lemma 6.1 imply that if $M_{p,q,\mathbf{l},\mathbf{w}}$ is smooth it must be a spin manifold. All the non-spin cases are orbifolds.

Excluding the homogeneous quadric $p = q = 2$, Collins and Székelyhidi [CS19] prove that the link $L_{p,q}$ has an SE metric if and only if $2p > q$ and $2q > p$. The slice that gives the possible SE metric is determined by $c_1(\mathcal{F}_\xi) = 4[d\eta]_B$.

Applying Theorem 3.3 gives SE metrics on the $S_{\mathbf{w}}^3$ joins $M_{p,q,\mathbf{l},\mathbf{w}} = L_{p,q} \star_{\mathbf{l}} S_{\mathbf{w}}^3$ where the components of \mathbf{l} are the relative Fano indices. Since these indices are fixed we relabel the join as $M_{p,q,\mathbf{w}}$. We have from Theorem 3.3 and Theorem 1.2 of [CS19]

Theorem 6.6. *If $2q > p$ and $2p > q$ there is an SE metric of complexity one in the \mathbf{w} Sasaki cone $\mathfrak{t}_{\mathbf{w}}^+$ of the join*

$$M_{p,q,\mathbf{w}} = k(S^2 \times S^3) \star_1 S_{\mathbf{w}}^3$$

where

$$l^0 = \frac{2(p+q)}{\gcd(2(p+q), |\mathbf{w}|)}, \quad l^\infty = \frac{|\mathbf{w}|}{\gcd(2(p+q), |\mathbf{w}|)}.$$

The converse of this theorem is not known at this time.

From Proposition 2.4 we see that the 7-manifolds $M_{p,q,\mathbf{w}} = k(S^2 \times S^3) \star_1 S_{\mathbf{w}}^3$ are simply connected with $\pi_2(M_{p,q,\mathbf{w}}) = \mathbb{Z}^{k+1}$.

6.3. Complexity Two Examples in Dimension 7. In this case $\mathfrak{t}_{\mathbf{w}}^+$ is the full Sasaki cone, i.e. the dimension of the Sasaki cone is two. Since any Sasakian structure on M is quasi-regular, there are many cases. Here it suffices to consider a simple example. Consider the Brieskorn-Pham link $L_{k,p}$ defined by

$$(57) \quad z_0^k + z_1^k + z_2^{k+1} + z_3^p = 0$$

of degree $pk(k+1)$ and weight vector $\mathbf{w} = ((k+1)p, (k+1)p, kp, k(k+1))$. The index of the natural quotient is

$$(58) \quad J_{k,p} = |\mathbf{w}| - d = 2pk + 2p + k - (p-1)k^2.$$

These hypersurfaces give infinitely many negative Sasakian structures, finitely many positive Sasakian structures, and one null Sasakian structure. First we only consider $k, p > 1$ since if k or p is one we obtain the standard 5 sphere. For simplicity we consider the case that p is relatively prime to both k and $k+1$ in which case $L_{k,p}$ is diffeomorphic to S^5 (see Example 10.3.12 in [BG08]). This eliminates the null case which has $(k, p) = (3, 12)$. It is easy to see that S^5 admits no null (Corollary 10.3.9 in [BG08]) nor indefinite Sasakian structures. We also note that $k = 2$ is a special case of the complexity one hypersurfaces considered in Section 6.2. So here we consider $k \geq 3$. It is easy to see that the induced Sasakian structure is negative if

$$k > 3, p > 3, \quad k = 3, p > 12, \quad k \geq 6, p = 2, 3, \quad \text{or} \quad k = 5, p = 3$$

and positive otherwise. So the only positive Sasakian structures on the hypersurfaces $L_{k,p} \approx S^5$ with $\gcd(k, p) = \gcd(k+1, p) = 1$ are $L_{3,5}, L_{3,7}, L_{3,11}, L_{4,3}$ all of which admit SE metrics, hence cscS metrics, which can be seen from [BGK05, BG08]. All negative Sasakian structures on links of weighted hypersurface singularities admit S- η -E metrics, hence cscS metrics, by the transverse Aubin-Yau Theorem [EKA90]. Summarizing we have

Theorem 6.7. *For all k, p such that p is relatively prime to k and $k+1$, and relatively prime pairs (l^0, l^∞) and (w^0, w^∞) , there exists a ray of constant scalar curvature Sasakian structures in $\mathfrak{t}_{\mathbf{w}}^+$ on the 7 dimensional Sasaki orbifolds $M_{k,p,1,\mathbf{w}}$, and the entire \mathbf{w} cone $\mathfrak{t}_{\mathbf{w}}^+$ is exhausted by extremal Sasaki metrics. Moreover, if l^∞ is large enough there are at least 3 cscS metrics in $\mathfrak{t}_{\mathbf{w}}^+$. $M_{k,p,1,\mathbf{w}}$ is a smooth manifold if and only if $\gcd(\text{lcm}(k, k+1, p)l^\infty, w^0 w^\infty l^0) = 1$.*

Remark 6.8. Again as in Remark 6.5 if $M_{k,p,1,\mathbf{w}}$ is smooth it must be a spin manifold. All non-spin cases are orbifolds.

We now have the analogue of Example 6.1 and Lemma 6.3

Lemma 6.9. *If $\gcd(k, p) = \gcd(k+1, p) = 1$, then the quotient orbifold $(N, \Delta_{k,p})$ generated by the natural Reeb vector field ξ_0 on the link $L_{k,p}$ is the log pair $(\mathbb{C}\mathbb{P}^2[k, 1, 1], \Delta_{k,p})$ with the branch divisor*

$$\Delta_{k,p} = \left(1 - \frac{1}{k+1}\right)(z_2 = 0) + \left(1 - \frac{1}{p}\right)(z_3 = 0).$$

Moreover, there is an algebraic quotient singularity with isotropy \mathbb{Z}_k occurring at the intersection of the irreducible components of $\Delta_{k,p}$, namely the points $[1, e^{\frac{i\pi(2m+1)}{k}}, 0, 0]$, $m = 0, \dots, k-1$.

Proof. The proof is similar to that of Lemma 6.3 and left as an exercise. \square

Note that the order of the link $L_{k,p}$ is $\text{lcm}(k, k+1, p)$.

6.4. Topology of the Joins. Clearly, we can apply our Theorems 3.2 and 3.3 to obtain many new cscS and SE metrics. A particularly interesting case is when M is a Sasaki homotopy sphere as studied, for example, in [BGN03, BGK05, GK07]. It is well known that there are precisely 28 oriented diffeomorphism types of homotopy spheres in dimension 7 and these all bound a parallelizable manifold. If we take the $S_{\mathbf{w}}^3$ join of these we obtain a simply connected Sasaki 9-manifold with $\pi_2 = \mathbb{Z}$ that bounds a parallelizable manifold. What are the possible diffeomorphism types? There are only 2 diffeomorphism types of homotopy spheres that bound a parallelizable manifold in dimension 9. Similarly there are many more oriented diffeomorphism types in dimension 11 than in dimension 9. Although it is beyond the scope of the present work, it would be very interesting to understand how the join operation relates to these diffeomorphism types.

We have already some elementary results in Proposition 2.4. Here we adopt the method described in [BTF16] to the general quasi-regular situation. The fibration (3) and Diagram (5) together with the torus bundle with total space $M \times S_{\mathbf{w}}^3$ gives the commutative diagram of fibrations

$$(59) \quad \begin{array}{ccccc} M \times S_{\mathbf{w}}^3 & \longrightarrow & M_{1,\mathbf{w}} & \longrightarrow & \mathbf{B}S^1 \\ \downarrow = & & \downarrow \pi & & \downarrow \psi \\ M \times S_{\mathbf{w}}^3 & \longrightarrow & \mathbf{B}(N, \Delta) \times \mathbf{B}\mathbb{C}\mathbb{P}^1[\mathbf{w}] & \longrightarrow & \mathbf{B}S^1 \times \mathbf{B}S^1 \end{array}$$

where $\mathbf{B}G$ is the classifying space of a group G or Haefliger's classifying space [Hae84] of an orbifold if G is an orbifold. The map ψ of Diagram (59) is that induced by the inclusion $e^{i\theta} \mapsto (e^{il^\infty\theta}, e^{-il^0\theta})$. Note that the lower fibration is a product of fibrations. The orbifold cohomology of $\mathbb{C}\mathbb{P}^1[\mathbf{w}]$ was computed in [BTF16]

$$H_{orb}^r(\mathbb{C}\mathbb{P}^1[\mathbf{w}], \mathbb{Z}) = H^r(\mathbf{B}\mathbb{C}\mathbb{P}^1[\mathbf{w}], \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } r = 0, 2, \\ \mathbb{Z}_{w^0 w^\infty} & \text{for } r > 2 \text{ even,} \\ 0 & \text{for } r \text{ odd.} \end{cases}$$

The method then consists of computing the differentials of the fibration

$$(60) \quad M \longrightarrow \mathbf{B}(N, \Delta) \longrightarrow \mathbf{B}S^1$$

and using the commutative diagram (59) to obtain the cohomology ring of $M_{1,\mathbf{w}}$. This method dates back to [WZ90] for the manifold case and was generalized to the $S_{\mathbf{w}}^3$ -join for $M = S^{2n+1}$ with its standard constant curvature metric in [BTF16], and in [BTF19] when M is a $Y^{p,q}$ with a quasi-regular SE metric. In both cases the torsion group $\mathbb{Z}_{w^0 w^\infty (l^0)^2}$ occurs in $H^4(M_{1,\mathbf{w}}, \mathbb{Z})$.

Proposition 6.10. *Let $M_{1,\mathbf{w}} = M \star_1 S_{\mathbf{w}}^3$ and assume that M has dimension greater than 3, is simply connected and that $b_3(M) = 0$. Then $H^4(M_{1,\mathbf{w}}, \mathbb{Z})$ contains the torsion group $\mathbb{Z}_{w^0 w^\infty (l^0)^2}$.*

Proof. Consider the differentials in the Leray-Serre spectral sequence of the bottom fibration of (59) which is a product. For the S^3 fibration we have $d_4(\alpha) = w^0 w^\infty s_2^2$ where α is the orientation class of S^3 and s_2 is the generating class of the second BS^1 . Now for the M fibration Proposition 2.4, $b_3(M) = 0$ and the Universal Coefficients Theorem imply that $H^3(M, \mathbb{Z}) = 0$. It follows that in the spectral sequence of the fibration (60) s_1^2 survives to E_∞ . By naturality the differentials in the bottom product fibrations of Diagram (59) pullback to the differentials of the top fibration, and since the diagram commutes we have $d_4(\alpha) = w^0 w^\infty (l^0)^2 s^2$. This class survives to E_∞ of the top fibration, thus, producing the torsion group in H^4 . \square

Further details depend on the differentials in the spectral sequence of the fibration (60).

Example 6.1. A particular case of interest in Lemma 6.3 is when $\gcd(p, q) = 1$ or equivalently $k = 0$ which gives $M = S^5$. However, we only need the manifold M to be homeomorphic to S^5 . In what follows we assume that $M_{1,\mathbf{w}} = S^5 \star_1 S_{\mathbf{w}}^3$. We have

Lemma 6.11. *$H^3(M_{1,\mathbf{w}}, \mathbb{Z}) = 0$ and $M_{1,\mathbf{w}}$ has the rational cohomology ring of $S^2 \times S^5$.*

Proof. From the Leray-Serre spectral sequence of the fibration

$$S^3 \times S^5 \longrightarrow M_{1,\mathbf{w}} \longrightarrow BS^1$$

we see that $E_2^{r,s} = E_4^{r,s}$. We let α be the orientation class of S^3 , then, over \mathbb{Q} , there are two possibilities for d_4 , either $d_4(\alpha) = s^2$ or $d_4(\alpha) = 0$ where s is the 2-class of BS^1 . If the latter holds then the 3-class α will survive to E_∞ , and there is only one such class. But by the Leray-Serre Theorem this converges to the cohomology of $M_{1,\mathbf{w}}$ which has a Sasaki metric. This implies that the rank of $H^3(M_{1,\mathbf{w}}, \mathbb{Q})$ must be even which gives a contradiction. Thus, we must have $d_4(\alpha) = s^2$, and this implies that $H^3(M_{1,\mathbf{w}}, \mathbb{Q}) = 0$. But from Proposition 2.4 we have $H_2(M_{1,\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}$, so H^3 vanishes over \mathbb{Z} as well by the Universal Coefficients Theorem. Then by naturality we have $d_4(\alpha \otimes s^j) = s^{j+1}$. So for $E^{2r,0}$ only $E^{2,0}$ survives to E_∞ . The only classes that survive to E^∞ are the 2-class s on the base, the 5-class β from the fiber, and the 7-class $\beta \otimes s$. \square

We now have

Proposition 6.12. *Consider the join $M_{1,\mathbf{w}} = S^5 \star_1 S_{\mathbf{w}}^3$ and assume also that $M_{1,\mathbf{w}}$ is a smooth manifold. Then the cohomology ring of $M_{1,\mathbf{w}}$ is*

$$\mathbb{Z}[x, y] / (w^0 w^\infty (l^0)^2 x^2, x^3, x^2 y, y^2)$$

where x is a 2-class and y is a 5-class.

Proof. As with Theorem 4.5 of [BTF16] we use naturality applied to the commutative diagram (59). In fact the result agrees with Theorem 4.5 although no assumption of regularity of the Reeb vector on S^5 is made. However, here we need to impose the smoothness condition; whereas, it is automatic in Theorem 4.5 of [BTF16]. As in Theorem 6.4 $M_{1,\mathbf{w}}$ is smooth if and only if $\gcd(2pql^\infty, w^0w^\infty l^0) = 1$. Now we know from Proposition 2.4 and Poincaré duality that $H^6(M_{1,\mathbf{w}}, \mathbb{Z}) = 0$ and $H^5(M_{1,\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}$. Moreover, from Lemma 6.11 $H^3(M_{1,\mathbf{w}}, \mathbb{Z}) = 0$ and from Proposition 6.10 $H^4(M_{1,\mathbf{w}}, \mathbb{Z})$ contains the torsion group $\mathbb{Z}_{w^0w^\infty(l^0)^2}$. One easily sees from the Leray-Serre spectral sequence of the upper fibration in Diagram 59 that $H^4(M_{1,\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}_{w^0w^\infty(l^0)^2}$ from which the cohomology ring easily follows. \square

Remark 6.13. This proposition generalizes to higher dimensional spheres as long as the smoothness condition of the join holds. That is, if $M_{1,\mathbf{w}} = S^{2r+1} \star_1 S_{\mathbf{w}}^3$ is a smooth manifold, then its cohomology ring is

$$\mathbb{Z}[x, y]/(w^0w^\infty(l^0)^2x^2, x^{r+1}, x^2y, y^2)$$

where x is a 2-class, and y is a $(2r + 1)$ -class.

Proposition 6.12 implies that the $S_{\mathbf{w}}^3$ join with any 5-sphere has the same cohomology ring. In particular, we apply this to the standard toric join, the complexity one links $L_{p,q}$ with $\gcd(p, q) = 1$ and the complexity two links $L_{k,p}$ with $\gcd(p, k) = \gcd(p, k + 1) = 1$. We are now ready to apply Sullivan's rational homotopy theory [Sul77]. As described in [WZ90] this implies that if a collection of simply connected closed manifolds have the same cohomology ring, the same rational Pontrjagin classes, and if the so-called minimal model is a formal consequence of the rational cohomology ring, then the collection has only finitely many diffeomorphism types. Let $\mathcal{C}_{p,q,1,\mathbf{w}}$ be the collection of smooth $S_{\mathbf{w}}^3$ -joins $L_{p,q} \star_1 S_{\mathbf{w}}^3$ of Brieskorn links with $w^0w^\infty(l^0)^2$ fixed and satisfy Equation (54) with $\gcd(p, q) = 1$. Similarly let $\mathcal{C}_{k,p,1,\mathbf{w}}$ be the collection of smooth $S_{\mathbf{w}}^3$ -joins $L_{k,p} \star_1 S_{\mathbf{w}}^3$ of Brieskorn links with $w^0w^\infty(l^0)^2$ fixed and satisfy Equation (57) with $\gcd(p, k) = \gcd(p, k + 1) = 1$. These collections have the cardinality of \mathbb{N} . Then we have

Proposition 6.14. *There are finitely many diffeomorphism types within the collection $\mathcal{C}_{p,q,1,\mathbf{w}}$. There are finitely many diffeomorphism types within the collection $\mathcal{C}_{k,p,1,\mathbf{w}}$ with fixed k .*

Proof. Since $w^0w^\infty(l^0)^2$ is a fixed odd number within each collection, the cohomology rings coincide. So according to Sullivan's theory we need only check that the first Pontrjagin class is fixed within each collection, and that the join is formal in each case. By Lemma 6.11 all the elements of each collection have the rational cohomology ring of $S^2 \times S^5$, so they are formal. To compute the Pontrjagin class we make note of the map π in the Commutative Diagram (59). Since the vertical bundle is trivial, $p_1(M_{1,\mathbf{w}})$ is the pullback of $p_1(\mathbb{B}(N, \Delta) \times \mathbb{B}\mathbb{C}\mathbb{P}^1[\mathbf{w}])$. This is well defined since the underlying topological space $(N, \Delta) \times \mathbb{C}\mathbb{P}^1$ is a combinatorial manifold determined up to combinatorial equivalence [Rv57, Tho58]. Then it follows from Lemma 6.3 that for the collection $\mathcal{C}_{p,q,1,\mathbf{w}}$ the underlying topological space is independent of p, q, l^∞ , so the first statement follows. In the case of the collection $\mathcal{C}_{k,p,1,\mathbf{w}}$ Lemma 6.9 tells us that the quotient singularity of the underlying topological space depends on k , so we should fix k for the underlying spaces to be combinatorially equivalent. \square

From Lemma 6.1 and Proposition 6.14 we obtain

Corollary 6.15. *For each odd natural number of the form $w^0w^\infty(l^0)^2$ there exists a simply connected smooth 7-manifold with infinitely many inequivalent contact structures of Sasaki type which admit cscS metrics.*

These results can be compared to a very similar result in our unpublished manuscript [BTF14b] in which the contact structures are toric; whereas, those above have complexity one or two.

REFERENCES

- [AC21] Vestislav Apostolov and David M. J. Calderbank, *The CR geometry of weighted extremal Kähler and Sasaki metrics*, Math. Ann. **379** (2021), no. 3-4, 1047–1088. MR 4238260
- [ACG06] Vestislav Apostolov, David M. J. Calderbank, and Paul Gauduchon, *Hamiltonian 2-forms in Kähler geometry. I. General theory*, J. Differential Geom. **73** (2006), no. 3, 359–412. MR 2228318 (2007b:53149)
- [ACGL20] Vestislav Apostolov, David M. J. Calderbank, Paul Gauduchon, and Eveline Legendre, *Levi-Kähler reduction of CR structures, products of spheres, and toric geometry*, Math. Res. Lett. **27** (2020), no. 6, 1565–1629. MR 4216597
- [ACGTF04] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, and C. W. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry. II. Global classification*, J. Differential Geom. **68** (2004), no. 2, 277–345. MR 2144249
- [ACGTF08] Vestislav Apostolov, David M. J. Calderbank, Paul Gauduchon, and Christina W. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry. III. Extremal metrics and stability*, Invent. Math. **173** (2008), no. 3, 547–601. MR MR2425136 (2009m:32043)
- [ACGTF11] ———, *Extremal Kähler metrics on projective bundles over a curve*, Adv. Math. **227** (2011), 2385–2424.
- [AH06] Klaus Altmann and Jürgen Hausen, *Polyhedral divisors and algebraic torus actions*, Math. Ann. **334** (2006), no. 3, 557–607. MR 2207875
- [Bel01] F. A. Belgun, *Normal CR structures on compact 3-manifolds*, Math. Z. **238** (2001), no. 3, 441–460. MR 2002k:32065
- [BG00a] Charles P. Boyer and Krzysztof Galicki, *A note on toric contact geometry*, J. Geom. Phys. **35** (2000), no. 4, 288–298. MR MR1780757 (2001h:53124)
- [BG00b] ———, *On Sasakian-Einstein geometry*, Internat. J. Math. **11** (2000), no. 7, 873–909. MR 1792957
- [BG08] ———, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008. MR MR2382957 (2009c:53058)
- [BGK05] C. P. Boyer, K. Galicki, and J. Kollár, *Einstein metrics on spheres*, Ann. of Math. (2) **162** (2005), no. 1, 557–580. MR 2178969 (2006j:53058)
- [BGN03] C. P. Boyer, K. Galicki, and M. Nakamaye, *Sasakian geometry, homotopy spheres and positive Ricci curvature*, Topology **42** (2003), no. 5, 981–1002. MR 1 978 045
- [BGO07] Charles P. Boyer, Krzysztof Galicki, and Liviu Ornea, *Constructions in Sasakian geometry*, Math. Z. **257** (2007), no. 4, 907–924. MR MR2342558 (2008m:53103)
- [BGS08] Charles P. Boyer, Krzysztof Galicki, and Santiago R. Simanca, *Canonical Sasakian metrics*, Commun. Math. Phys. **279** (2008), no. 3, 705–733. MR MR2386725
- [BHLTF18] Charles P. Boyer, Hongnian Huang, Eveline Legendre, and Christina W. Tønnesen-Friedman, *Reducibility in Sasakian geometry*, Trans. Amer. Math. Soc. **370** (2018), no. 10, 6825–6869. MR 3841834
- [BHLTF21] Charles P. Boyer, Hongnian Huang, Eveline Legendre, and Christina W. Tønnesen-Friedman, *Some open problems in Sasaki geometry*, Differential Geometry in the Large, Lecture Note Series, vol. 463, London Mathematical Society, Cambridge University Press, 2021, pp. 143–168.

- [BM93] A. Banyaga and P. Molino, *Géométrie des formes de contact complètement intégrables de type toriques*, Séminaire Gaston Darboux de Géométrie et Topologie Différentielle, 1991–1992 (Montpellier), Univ. Montpellier II, Montpellier, 1993, pp. 1–25. MR 94e:53029
- [Boy19] Charles P. Boyer, *Contact structures of Sasaki type and their associated moduli*, Complex Manifolds **6** (2019), no. 1, 1–30. MR 3912431
- [BP14] Charles P. Boyer and Justin Pati, *On the equivalence problem for toric contact structures on S^3 -bundles over S^2* , Pacific J. Math. **267** (2014), no. 2, 277–324. MR 3207586
- [BT82] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York, 1982. MR 658304 (83i:57016)
- [BTF14a] Charles P. Boyer and Christina W. Tønnesen-Friedman, *Extremal Sasakian geometry on S^3 -bundles over Riemann surfaces*, Int. Math. Res. Not. IMRN (2014), no. 20, 5510–5562. MR 3271180
- [BTF14b] ———, *Simply connected manifolds with infinitely many toric contact structures and constant scalar curvature Sasaki metrics*, preprint; arXiv:1404.3999 (2014).
- [BTF15] ———, *The Sasaki join and admissible Kähler constructions*, J. Geom. Phys. **91** (2015), 29–39. MR 3327046
- [BTF16] ———, *The Sasaki join, Hamiltonian 2-forms, and constant scalar curvature*, J. Geom. Anal. **26** (2016), no. 2, 1023–1060. MR 3472828
- [BTF19] ———, *Sasaki-Einstein metrics on a class of 7-manifolds*, J. Geom. Phys. **140** (2019), 111–124. MR 3923473
- [BTF20a] ———, *Iterated S^3 Sasaki joins and Bott orbifolds*, arXiv:2006.06596, to appear in Annales Scientifiques de la Faculté de Toulouse. (2020).
- [BTF20b] ———, *On positivity in Sasaki geometry*, Geom. Dedicata **204** (2020), 149–164. MR 4056696
- [BvC18] Charles P. Boyer and Craig van Coevering, *Relative K-stability and extremal Sasaki metrics*, Math. Res. Lett. **25** (2018), no. 1, 1–19.
- [CS18] Tristan C. Collins and Gábor Székelyhidi, *K-semistability for irregular Sasakian manifolds*, J. Differential Geom. **109** (2018), no. 1, 81–109. MR 3798716
- [CS19] Tristan C. Collins and Gábor Székelyhidi, *Sasaki-Einstein metrics and K-stability*, Geom. Topol. **23** (2019), no. 3, 1339–1413. MR 3956894
- [Don02] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), no. 2, 289–349. MR MR1988506 (2005c:32028)
- [EKA90] A. El Kacimi-Alaoui, *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math. **73** (1990), no. 1, 57–106. MR 91f:58089
- [Gei97] H. Geiges, *Normal contact structures on 3-manifolds*, Tohoku Math. J. (2) **49** (1997), no. 3, 415–422. MR 98h:53046
- [GK07] Alessandro Ghigi and János Kollár, *Kähler-Einstein metrics on orbifolds and Einstein metrics on spheres*, Comment. Math. Helv. **82** (2007), no. 4, 877–902. MR MR2341843 (2008j:32027)
- [Gui94] V. Guillemin, *Kähler structures on toric varieties*, J. Differential Geom. **40** (1994), no. 2, 285–309. MR 1293656 (95h:32029)
- [Hae84] A. Haefliger, *Groupoïdes d’holonomie et classifiants*, Astérisque (1984), no. 116, 70–97, Transversal structure of foliations (Toulouse, 1982). MR 86c:57026a
- [HS15] Weiyong He and Song Sun, *The generalized Frankel conjecture in Sasaki geometry*, Int. Math. Res. Not. IMRN (2015), no. 1, 99–118. MR 3340296
- [Leg11a] Eveline Legendre, *Existence and non-uniqueness of constant scalar curvature toric Sasaki metrics*, Compos. Math. **147** (2011), no. 5, 1613–1634. MR 2834736
- [Leg11b] ———, *Toric geometry of convex quadrilaterals*, J. Symplectic Geom. **9** (2011), no. 3, 343–385. MR 2817779 (2012g:53079)
- [Leg16] ———, *Toric Kähler-Einstein metrics and convex compact polytopes*, J. Geom. Anal. **26** (2016), no. 1, 399–427. MR 3441521
- [Ler02] E. Lerman, *Contact toric manifolds*, J. Symplectic Geom. **1** (2002), no. 4, 785–828. MR 2039164

- [Ler04] ———, *Homotopy groups of K -contact toric manifolds*, Trans. Amer. Math. Soc. **356** (2004), no. 10, 4075–4083 (electronic). MR 2 058 839
- [MSY08] Dario Martelli, James Sparks, and Shing-Tung Yau, *Sasaki-Einstein manifolds and volume minimisation*, Comm. Math. Phys. **280** (2008), no. 3, 611–673. MR MR2399609 (2009d:53054)
- [RT11] Julius Ross and Richard Thomas, *Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics*, J. Differential Geom. **88** (2011), no. 1, 109–159. MR 2819757
- [Rv57] V. A. Rohlin and A. S. Švarc, *The combinatorial invariance of Pontrjagin classes*, Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 490–493. MR 0102070
- [Sul77] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269–331 (1978). MR 0646078 (58 #31119)
- [Tho58] R. Thom, *Les classes caractéristiques de Pontrjagin des variétés triangulées*, Symposium internacional de topología algebraica International symposium on algebraic topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 54–67. MR 0102071
- [Tia97] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), no. 1, 1–37. MR 99e:53065
- [WZ90] M. Y. Wang and W. Ziller, *Einstein metrics on principal torus bundles*, J. Differential Geom. **31** (1990), no. 1, 215–248. MR 91f:53041

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