SASAKIAN MANIFOLDS WITH PERFECT FUNDAMENTAL GROUPS

CHARLES P. BOYER AND CHRISTINA W. TØNNESEN-FRIEDMAN

ABSTRACT. Using the Sasakian join construction with homology 3-spheres, we give a countably infinite number of examples of Sasakian manifolds with perfect fundamental group in all odd dimensions ≥ 3. These have extremal Sasaki metrics with constant scalar curvature. Moreover, we present further examples of both Sasaki-Einstein and Sasaki-η-Einstein metrics.

Dedicated to Professor Augustin Banyaga on the occasion of his 65th birthday

1. INTRODUCTION

Until recently, except for the rather obvious examples, little seemed to be known about Sasakian manifolds with non-trivial fundamental group; see, however, [Che11]. In this note we construct many examples of Sasakian manifolds with a perfect fundamental group. The examples we present all have extremal Sasaki metrics (in the sense of Calabi), and have constant scalar curvature. When the Sasaki cone has dimension greater than one, the Openness Theorem of [BGS08] implies the existence of other extremal Sasaki metrics which generally do not have constant scalar curvature. We leave for future work the explicit construction of extremal non constant scalar curvature Sasaki metrics. Our main purpose here is to prove the following theorems:

Theorem 1.1. *For each odd dimension ≥ 3 there exists a countable infinity of Sasakian manifolds with a perfect fundamentfinial group which admit Sasaki metrics with constant scalar curvature. Furthermore, there is an infinite number of such Sasakian manifolds that have the integral cohomology ring of $S^2 \times S^{2r+1}$.*

Theorem 1.2. *There exist a countably infinite number of aspherical contact 5-manifolds with perfect fundamental group and the integral cohomology ring of $S^2 \times S^3$ that admit Sasaki metrics with constant scalar curvature. Moreover, there are such manifolds that admit a ray of Sasaki-η-Einstein metrics (hence, Lorentzian Sasaki-Einstein metrics).*

The manifolds of Theorem 1.2 differ from those of Theorem 1.1 with $r = 1$ which are not aspherical since they have $\pi_3 = \mathbb{Z}$.

Theorem 1.3. *There exist negative Sasaki-η-Einstein (hence, Lorentzian Sasaki-Einstein) 7-manifolds with perfect fundamental group and arbitrary second Betti number ≥ 1.*

Theorem 1.4. *Let N be a simply connected quasi-regular Sasaki-Einstein $(2r + 1)$-dimensional manifold with Fano index I_F and order v. Then if $\gcd(30 I_F, v) = 1$, the join $S^3/\mathbb{I}^* \star_{1, I_F} N$ is a smooth $(2r + 3)$-manifold with perfect fundamental group and admits a Sasaki-Einstein metric. Such examples exist with rational cohomology ring of $S^2 \times S^{2r+1}$ for all $r \geq 1$. Moreover, $S^3/\mathbb{I}^* \star_{1,2} S^3$ is Sasaki-Einstein and has integral cohomology ring of $S^2 \times S^3$.*

Many other examples of both Sasaki-Einstein and Sasaki-η-Einstein manifolds with perfect fundamental group can be worked out by choosing N appropriately, as for example in Chapters 11 and 13 of [BG08], and in particular the 5-dimensional Sasaki-Einstein manifolds in [BGK05, Kol05, Kol07, BN10].

After presenting the necessary foundations in Sections 2 and 3, the proofs of Theorems 1.1-1.4 are given, respectively, in Subsections 5.1-5.4. The topological parts are presented in Section 4, and the geometrical parts in Section 5.

2. Preliminaries on Sasakian Geometry

Here we give a brief review of Sasakian geometry referring to [BG08] for details and further development. Sasakian geometry can be thought of roughly as the odd dimensional version of Kählerian geometry. Its relation to contact geometry mimics the relation of Kähler geometry to symplectic geometry.

2.1. **Sasakian Structures.** A Sasakian structure on a smooth manifold M of dimension $2n + 1$ consists of a contact 1-form η together with its Reeb vector field ξ which satisfies $\eta(\xi) = 1$ and $\xi \lrcorner d\eta = 0$, an endomorphism field Φ which annihilates ξ and defines a strictly pseudoconvex CR structure $(\mathcal{D}, J)$ satisfying $\mathcal{D} = \ker \eta$ and $J = \Phi|_{\mathcal{D}}$, and

finally a compatible Riemannian metric g defined by the equation

$$(1) \qquad\qquad g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta,$$

such that ξ is a Killing vector field of g. We denote such a Sasakian structure by the quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$. Note that the Reeb vector field ξ generates a one dimensional foliation $\mathcal{F}_\xi$ of M whose transverse structure is Kähler with transverse Kähler form $d\eta$. The transverse Kähler metric on $\mathcal{D}$ is $g_T = d\eta \circ (\Phi \otimes \mathbb{1})$. There is a freedom of scaling, namely, given a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ consider the *transverse homothety* defined by sending the Sasakian structure to

$$(2)$$
$$\mathcal{S} = (\xi, \eta, \Phi, g) \mapsto \mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a) \text{ where } a \in \mathbb{R}^+ \text{ and } g_a = ag + (a^2 - a)\eta \otimes \eta.$$

$\mathcal{S}_a$ is another Sasakian structure which generally is inequivalent to $\mathcal{S}$; hence, Sasakian structures come in rays.

When M is compact the closure of any leaf of $\mathcal{F}_\xi$ is a torus $\mathfrak{T}$ of dimension at least one, and the flow is conjugate to a linear flow on the torus (cf. Theorem 2.6.4 of [BG08]). This implies that for a dense subset of Sasakian structures $\mathcal{S}$ on a compact manifold the leaves are all compact 1-dimensional manifolds, i.e circles. Such $\mathcal{S}$ are known as *quasiregular* in which case the foliation $\mathcal{F}_\xi$ comes from a locally free circle action. Assuming this circle action is effective the isotropy subgroups are all finite cyclic groups, and the least common multiple $\upsilon = \upsilon(\mathcal{S})$ of theirs orders is an invariant of the Sasakian structure $\mathcal{S}$, called the *order* of $\mathcal{S}$. Then the quotient space $\mathcal{Z}$ has the structure of a projective algebraic orbifold with an induced Kähler form ω such that $\pi^*\omega = d\eta$ where π is the quotient projection. The isotropy subgroups of the local circle action on M give rise to the local uniformizing groups of the orbifold. If the circles comprising the leaves of $\mathcal{F}_\xi$ all have the same period, $\mathcal{S}$ is said to be *regular*, and the quotient space $\mathcal{Z}$ is a smooth projective algebraic variety with a trivial orbifold structure. A leaf of $\mathcal{F}_\xi$ that is not quasi-regular is a copy of $\mathbb{R}$ in which case $\mathcal{F}_\xi$ is said to be *irregular*. A theorem of Rukimbira [Ruk95] says that an irregular Sasakian structure can be approximated by quasi-regular ones. A somewhat more general case occurs if we drop the condition that $(\mathcal{D}, J)$ be a CR structure, but only consider a strictly pseudoconvex *almost* CR structure, that is, the almost complex structure J is not necessarily integrable. Then $\mathcal{S} = (\xi, \eta, \Phi, g)$ is called a *K-contact* structure. A quasi-regular contact structure is equivalent to having a compatible K-contact structure [Ruk95]. All K-contact structures considered in this paper are Sasakian. The (almost) complex structure $\hat{J}$ on $\mathcal{Z}$ is also related to the (almost) CR structure J on M. For any foliate vector field X on

M we have $\pi_*\Phi X = \hat{J}\pi_* X$. We say that $J = \Phi|_{\mathcal{D}}$ is the *horizontal lift* of $\hat{J}$.

The flow of the Reeb vector field ξ lies in the center of the automorphism group $\mathfrak{Aut}(\mathcal{S})$ of a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$; hence, any Sasakian structure on a compact $(2n+1)$-dimensional manifold has a k-dimensional torus $\mathfrak{T}_k$ in its automorphism group, where $1 \le k \le n+1$. The subset $\mathfrak{t}_k^+$ of the Lie algebra $\mathfrak{t}_k$ of this torus consisting of vector fields ξ' that satisfy the positivity condition $\eta(\xi') > 0$ everywhere on M forms a cone called the *Sasaki cone* [BGS08]. It provides a k-dimensional family of Sasakian structures associated with $\mathcal{S}$ all having the same underlying CR structure $(\mathcal{D}, J)$. There can be many Sasaki cones associated with the same contact structure $\mathcal{D}$ as seen, for example, in [Boy10, Boy11, BTF11, BTF12]. These give rise to bouquets of Sasaki cones which correspond to distinct conjugacy classes of tori in the contactomorphism group $\mathfrak{Con}(\mathcal{D})$. For more on the important infinite dimensional Fréchet Lie group $\mathfrak{Con}(\mathcal{D})$ we refer to Banyaga's seminal book [Ban97].

The conditions on the Riemannian curvature for Sasaki metrics have been very well studied, and we refer to [Bla10, BG08] and references therein for details. It suffices here to mention only some basic facts about the Ricci curvature and Φ sectional curvature of a Sasaki metric. Let $\mathcal{S} = (\xi, \eta, \Phi, g)$ be a Sasakian structure on a $(2n + 1)$-dimensional manifold, then the Ricci curvature Ric_g of g satisfies the identities

$$(3) \qquad \xi \lrcorner \mathrm{Ric}_g = 2n\eta, \qquad \mathrm{Ric}_g|_{\mathcal{D}\times\mathcal{D}} = \mathrm{Ric}_T - 2g|_{\mathcal{D}\times\mathcal{D}}$$

where Ric_T denotes the Ricci curvature of the transverse Kähler metric. One easily sees from these equations that the corresponding scalar curvatures are related by

$$(4) \qquad\qquad\qquad s_g = s_T - 2n.$$

If K is the ordinary Riemannian sectional curvature of g, then the Φ sectional curvature of g is defined by $K(X, \Phi X)$ where X is any vector field. Sasaki metrics of constant Φ sectional curvature are known as *Sasakian space forms*. There are three types of Sasakian space forms, those with $K(X, \Phi X) = c > -3, < -3$ and $= -3$. Within each type different constants are related by a transverse homothety. If we have a Sasakian space form of constant Φ sectional curvature c, then after a transverse homothety Equation (2), we obtain a Sasakian space form of constant Φ sectional curvature $c' = \frac{c+3}{a} - 3$. They are the analogs of constant holomorphic sectional curvature in complex geometry. Indeed, under the Boothby-Wang correspondence constant Φ sectional curvature c corresponds precisely to constant holomorphic transverse

sectional curvature $k = c + 3$. Assuming the Sasakian manifold is simply connected, in the spherical or positive case ($c > -3$) the transverse Kähler structure is that of complex projective space $\mathbb{CP}^n$; whereas, in the hyperbolic or negative case ($c < -3$) the transverse Kähler structure is that of the complex hyperbolic ball $\mathbb{B}^n$. Finally we make note of the Ricci tensor for the transverse Kähler structures of constant holomorphic sectional curvature k, viz.

$$(5) \qquad\qquad \mathrm{Ric}_T = \frac{n+1}{2} k g_T.$$

2.2. **The Join Construction.** We shall make use of the *join construction* first introduced in [BG00] in the context of Sasaki-Einstein manifolds, and developed further for general Sasakian structures in [BGO07], see also Section 7.6.2 of [BG08]. Products of Kählerian manifolds are Kähler, but products of Sasakian manifolds do not even have the correct dimension. Nevertheless, one can easily construct new quasi-regular Sasakian manifolds from old quasi-regular ones by constructing circle orbibundles over the product of Kähler orbifolds. Let M_i for $i = 1, 2$ be compact quasi-regular contact manifolds with Reeb vector fields ξ_i, respectively. These vector fields generate locally free circle actions on M_i and their quotients are symplectic orbifolds $\mathcal{Z}_i$. Then the quotient of the product $T^2 = S^1 \times S^1$ action on $M_1 \times M_2$ is $\mathcal{Z}_1 \times \mathcal{Z}_2$. Taking primitive symplectic forms ω_i on $\mathcal{Z}_i$ we consider the symplectic form $\omega_{k_1,k_2} = k_1 \omega_1 + k_2 \omega_2$ on $\mathcal{Z}_1 \times \mathcal{Z}_2$ where k_1, k_2 are relatively prime positive integers. Then by the orbifold Boothby-Wang construction [BG00] the total space of the principal circle orbibundle over $\mathcal{Z}_1 \times \mathcal{Z}_2$ corresponding to the cohomology class $[\omega_{k_1,k_2}] \in H^2(\mathcal{Z}_1 \times \mathcal{Z}_2, \mathbb{Z})$ has a natural quasi-regular contact structure whose contact form η_{k_1,k_2} satisfies $d\eta_{k_1,k_2} = \pi^* \omega_{k_1,k_2}$ where π is the natural orbibundle projection. Moreover, if the base spaces $\mathcal{Z}_i$ are complex orbifolds and the ω_i Kähler forms, the total space of this orbibundle, denoted by $M_1 \star_{k_1,k_2} M_2$, has a natural Sasakian structure. It is called *the join* of M_1 and M_2. Generally, $M_1 \star_{k_1,k_2} M_2$ is only an orbifold; however, if $\gcd(v_1 k_2, v_2 k_1) = 1$ it will be a smooth manifold, where v_i is the order the Sasakian structures on M_i. So if M_i has Sasakian structures $\mathcal{S}_i = (\xi_i, \eta_i, \Phi_i, g_i)$ we obtain a new Sasakian structure $\mathcal{S}_{k_1,k_2}$ on $M_1 \star_{k_1,k_2} M_2$ such that the

following diagram

$$M_1 \times M_2$$

(6)
$$\Big\downarrow \pi_B \qquad\qquad M_1 \star_{k_1,k_2} M_2$$

$$\mathcal{Z}_1 \times \mathcal{Z}_2$$

commutes. Here π_B is the quotient projection of the T^2 torus action, the southeast arrow is the quotient projection by the circle action generated by the vector field $\frac{1}{2k_1}\xi_1 - \frac{1}{2k_2}\xi_2$ and the southwest arrow is the quotient projection of the Reeb vector field $\frac{1}{2k_1}\xi_1 + \frac{1}{2k_2}\xi_2$ of $M_1 \star_{k_1,k_2} M_2$.

Although we are not able to distinguish diffeomorphism types of $M_1 \star_{k_1,k_2} M_2$ generally, we can distinguish contact structures by the first Chern class $c_1(\mathcal{D})$ of the contact bundle $\mathcal{D}$.

2.3. Extremal Sasakian Structures.
Given a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on a compact manifold M^{2n+1} we deform the contact 1-form by $\eta \mapsto \eta(t) = \eta + t\zeta$ where ζ is a basic 1-form with respect to the characteristic foliation $\mathcal{F}_\xi$ defined by the Reeb vector field ξ. Here t lies in a suitable interval containing 0 and such that $\eta(t) \wedge d\eta(t) \neq 0$. This gives rise to a family of Sasakian structures $\mathcal{S}(t) = (\xi, \eta(t), \Phi(t), g(t))$ that we denote by $\mathcal{S}(\xi, \bar{J})$ where $\bar{J}$ is the induced complex structure on the normal bundle $\nu(\mathcal{F}_\xi) = TM/L_\xi$ to the Reeb foliation $\mathcal{F}_\xi$ which satisfy the initial condition $\mathcal{S}(0) = \mathcal{S}$. On the space $\mathcal{S}(\xi, \bar{J})$ we consider the "energy functional" $E : \mathcal{S}(\xi, \bar{J}) \longrightarrow \mathbb{R}$ defined by

(7)
$$E(g) = \int_M s_g^2 d\mu_g,$$

i.e. the L^2-norm of the scalar curvature s_g of the Sasaki metric g. Critical points g of this functional are called *extremal Sasaki metrics*. Similar to the Kählerian case, the Euler-Lagrange equations for this functional says [BGS08] that g is critical if and only if the gradient vector field $J\mathrm{grad}_g s_g$ is transversely holomorphic, so, in particular, Sasakian metrics with constant scalar curvature are extremal. Since the scalar curvature s_g is related to the transverse scalar curvature s_g^T of the transverse Kähler metric by $s_g = s_g^T - 2n$, a Sasaki metric is extremal if and only if its transverse Kähler metric is extremal. Hence, in the quasi-regular case, an extremal Kähler orbifold metric lifts to an extremal Sasaki metric, and conversely an extremal Sasaki metric projects to an extremal Kähler orbifold metric. Note that the deformation $\eta \mapsto \eta(t) = \eta + t\zeta$ not only deforms the contact form, but also

deforms the contact structure $\mathcal{D}$ to an equivalent, isotopic, contact structure. So when we say that a contact structure $\mathcal{D}$ has an extremal representative, we mean so up to isotopy. Deforming the Kähler form within its Kähler class corresponds to deforming the contact structure within its isotopy class.

As mentioned above Sasaki metrics of constant scalar curvature are a special case of extremal Sasaki metrics. We shall abbreviate constant scalar curvature by CSC. A further special case of interest are the so-called Sasaki-η-Einstein metrics, or simply η-Einstein (see for example [BGM06, BG08] and references therein). Recall that a Sasakian (or K-contact) structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is called η-*Einstein* if there are constants a, b such that

$$(8) \qquad\qquad \mathrm{Ric}_g = ag + b\eta \otimes \eta$$

where Ric_g is the Ricci curvature of g. The constants a, b satisfy $a + b = 2n$. The scalar curvature s_g of an η-Einstein metric is constant. Indeed, if the manifold has dimension $2n+1$, then the scalar curvature satisfies $s_g = 2n(a + 1)$. However, as we can easily see, not every constant scalar curvature Sasaki metric is η-Einstein. In fact, many of the CSC Sasaki metrics that we construct have a diagonal Ricci tensor consisting of constant diagonal blocks. Notice that if $b = 0$ we obtain the more familiar Einstein metric, so η-Einstein is a generalization of Einstein. In this case the scalar curvature $s_g = 2n(2n + 1)$ and the transverse scalar curvature is $s_T = 4n(n + 1)$. Moreover, it follows from Equation (3) that a Sasaki metric g is η-Einstein if and only if the transverse Kähler metric h is Einstein. So one easily sees that a transverse Kähler-Einstein metric is negative if and only if $a < -2$, and positive if and only if $a > -2$. We refer to these as negative (positive) η-Einstein metrics, respectively. Furthermore, given a positive η-Einstein metric there is a transverse homothety whose resulting metric is Sasaki-Einstein. Even more is true in dimension 3: a 3-dimensional Sasaki metric is η-Einstein if and only if it has constant Φ sectional curvature [BGM06].

It is easy to see that any Sasakian structure gives rise naturally to a Sasaki metric with a Lorentzian signature, see Section 11.8.1 in [BG08]. Moreover, in the Lorentzian signature, one can apply a transverse homothety to a negative Sasaki-η-Einstein metric to obtain a Lorentzian Sasaki-Einstein metric. Thus, we can obtain many examples of Lorentzian Sasaki-Einstein metrics [Gom11].

3. Seifert Fibered Homology 3-Spheres

Homology spheres are by definition manifolds whose integral homology coincides with that of a sphere. In dimension 3 this is equivalent to the fundamental group being *perfect*, that is, it coincides with its commutator subgroup. We want the homology 3-spheres that we consider to admit a Sasakian structure; hence, they must be Seifert fibered homology spheres with an effective fixed point free circle action. Here we give a brief review of such homology 3-spheres following [Sav02, LR10] and the translation of [Sei33] in [ST80]. It is known that the binary icosahedral group $\mathbb{I}^*$ is the only non-trivial finite perfect subgroup of $SU(2)$ (see page 181 in [Wol67]). It is a double cover of the simple group $\mathbb{I}$ of order 60, the icosahedral group. Moreover, it follows from Perelman's proof of the Poincaré conjecture that up to diffeomorphism the only compact 3-manifold with a non-trivial finite perfect fundamental group is the celebrated Poincaré sphere $S^3/\mathbb{I}^*$. The remainder of the Seifert fibered homology 3-spheres, except for S^3, can be realized as a homogeneous space of the form $\widetilde{PSL}(2,\mathbb{R})/\Gamma$ where $\widetilde{PSL}(2,\mathbb{R})$ denotes the universal cover of the projective linear group $PSL(2,\mathbb{R})$, and Γ is a cocompact discrete subgroup of $\widetilde{PSL}(2,\mathbb{R})$ [NR78, RV81, LR10]. Hence, $\pi_1(M^3)$ is infinite and M^3 is aspherical. Recall that a manifold M is *aspherical* if $\pi_k(M) = 0$ for all $k > 1$. Summarizing we have

Proposition 3.1. *Let M^3 be a Seifert fibered homology 3-sphere which is not the standard sphere nor the Poincaré sphere. Then M^3 is aspherical and a homogeneous space of the form $\widetilde{\mathrm{PSL}}(2,\mathbb{R})/\Gamma$ for some cocompact infinite discrete subgroup Γ. Furthermore, $\pi_1(M^3)$ is infinite and perfect.*

3.1. **The Orbifold Base.** We consider the homology 3-sphere as the Seifert bundle $S^1 \longrightarrow M^3 \longrightarrow B$ where B is an orbifold. As will become evident below in all cases the base of the Seifert fibration is S^2 with an orbifold structure. We refer the reader to Chapter 4 of [BG08] for the basics of orbifolds. In order to work with orbifold cohomology classes we consider Haefliger's [Hae84] classifying space $\mathsf{B}B$ of an orbifold B. We represent the orbifold B by a proper étale Lie groupoid $\mathcal{G}$ (or any Morita equivalent Lie groupoid) and let $\mathsf{B}B$ denote the classifying space of $\mathcal{G}$, then the integral orbifold cohomology groups and homotopy groups are defined by $H^i_{orb}(B,\mathbb{Z}) = H^i(\mathsf{B}B,\mathbb{Z})$ and $\pi^{orb}_i(B,\mathbb{Z}) = \pi_i(\mathsf{B}B,\mathbb{Z})$, see Section 4.3 of [BG08] for further details.
   We have

Lemma 3.2. *Let M^3 be a Seifert fibered homology 3-sphere that is not the standard sphere S^3. Then $\pi^{orb}_1(B)$ is a quotient of $\pi_1(M^3)$, and*

hence, perfect. If M^3 is not the Poincaré sphere, then $\pi_i^{orb}(B) = 0$ for $i \geq 2$ and $\pi_1^{orb}(B) \approx \pi_1(M^3)/\mathbb{Z}$. If $M^3 = S^3/\mathbb{I}^*$, then $\pi_2^{orb}(B) \approx \mathbb{Z}$, $\pi_i^{orb}(B) \approx \pi_i(S^3/\mathbb{I}^*) \approx \pi_i(S^3)$ for $i \geq 3$, and $\pi_1^{orb}(B)$ is $\mathbb{I}$.

Proof. Since in all cases $\pi_2(M^3) = 0$ we have the homotopy exact sequence

$$(9) \qquad 0 \longrightarrow \pi_2^{orb}(B) \overset{\iota}{\longrightarrow} \mathbb{Z} \overset{\psi}{\longrightarrow} \pi_1(M^3) \longrightarrow \pi_1^{orb}(B) \longrightarrow \{1\}$$

which proves the first statement. If M^3 is not the Poincaré sphere then $\pi_1(M^3)$ is infinite, so by Lemma 14.3.1 of [LR10] ψ is injective which proves the second statement. If M^3 is the Poincaré sphere then $\pi_1(M^3) \approx \mathbb{I}^*$ and Lemma 14.3.1 of [LR10] says that B is S^2. From the exact sequence (9) and the fact that $\pi_1(M^3) = \mathbb{I}^*$ we see that $\pi_1^{orb}(B)$ is either $\mathbb{I}^*$ or $\mathbb{I}$. We claim that it must be $\mathbb{I}$. The following argument is taken from [Zim11]. The orbifold structure of B is described by the branched cover $S^2 \longrightarrow S^2/G$ where G is either $\mathbb{I}$ or $\mathbb{I}^*$. Now as we shall see shortly the Poincaré sphere can be represented by the link $L(2, 3, 5)$ and the orbifold Riemann-Hurwitz formula is

$$\chi(S^2) = |G|\big(2 - 2g - \sum_i (1 - \frac{1}{m_i})\big)$$

where χ is Euler characteristic and g is the genus of the Riemann surface, so we have $\chi(S^2) = 2$, $g = 0$, and $(m_1, m_2, m_3) = (2, 3, 5)$. Thus, the formula gives $|G| = 60$, so $G = \mathbb{I}$. Then from the exact sequence (9) we see that $\pi_2^{orb}(B) \approx \mathbb{Z}$ and the map ι is multiplication by 2. □

In all cases in this paper the 2-dimensional orbifold B is *developable*, that is, it is a global quotient, namely, $B = S^2/\pi_1^{orb}(B)$.

A result of Neumann and Raymond [NR78] says that up to orientation any Seifert fibered homology 3-sphere can be realized as a link of complete intersections of generalized Brieskorn manifolds [Ran75]. Moreover, it follows from Section 9.6 of [BG08] that any such link admits a Sasakian structure. Seifert fibrations as complete intersections is also treated in [Sav02, Loo84] as well as recently in Section 14.11 of [LR10].

3.2. **Complete Intersections of Brieskorn Manifolds.**
Here we essentially follow Section 14.11 of [LR10]. Let $\mathbf{a} = (a_0, \cdots, a_n)$ be a sequence of integers with $a_i \geq 2$ and let $C = (c_{ij})$ be an $n - 1$ by $n + 1$ matrix of complex numbers. We also assume that each $n - 1$ by $n - 1$ minor determinant of C is nonzero. Then the complex variety

(10)
$$V_C(\mathbf{a}) = \{z \in \mathbb{C}^{n+1} \mid f_i = c_{i0}z_0^{a_0} + \cdots + c_{in}z_n^{a_n} = 0, \; i = 0, \cdots, n - 2\}$$

is nonsingular away from the origin in $\mathbb{C}^{n+1}$, and the link

$$(11) \qquad\qquad L(\mathbf{a}) = V_C(\mathbf{a}) \cap S^{2n+1}$$

is a smooth 3-dimensional manifold which is independent of C up to diffeomorphism. Note that V_C has a $\mathbb{C}^*$ action defined by $z \mapsto (\lambda^{w_0} z_0, \cdots, \lambda^{w_n} z_n)$ with weights

$$w_j = \frac{\mathrm{lcm}(a_0, \cdots, a_n)}{a_j}.$$

The unit circle $S^1 \subset \mathbb{C}^*$ acts on the link $L(\mathbf{a})$ without fixed points, so it is a Seifert manifold. The Seifert invariants are given on page 336 of [LR10]. The unnormalized invariants are

$$M = \{\mathfrak{o}, g, 0, 0, s_1(\alpha_0, \beta_0), \ldots, s_n(\alpha_n, \beta_n)\}$$

where

$$\alpha_j = \frac{\mathrm{lcm}(a_0, \cdots, a_n)}{\mathrm{lcm}(a_0, \cdots, \widehat{a}_j, \cdots, a_n)}, \qquad s_j = \frac{a_0 a_1 \cdots \widehat{a}_j \cdots a_n}{\mathrm{lcm}(a_0, \cdots, \widehat{a}_j, \cdots, a_n)}$$

$$g = \frac{1}{2}\left(2 + (n-1)\frac{a_0 a_1 \cdots a_n}{\mathrm{lcm}(a_0, \cdots, a_n)} - \sum_{j=0}^{n} s_j\right)$$

and β_j and e are determined by

$$-e(M) = \sum_{j=0}^{n} s_j \frac{\beta_j}{\alpha_j} = \frac{a_0 a_1 \cdots a_n}{\mathrm{lcm}(a_0, \cdots, a_n)^2}.$$

This last equation becomes

$$(12) \qquad\qquad \sum_{j=0}^{n} \beta_j w_j = 1.$$

Here g is the genus of the base Riemann surface. Note that the variety V_C is not the most general type of complete intersection even for dimension 3. Generally, complete intersections have a multidegree $\mathbf{d} = (d_0, \cdots, d_{n-2})$, but in our case here the degrees d_i are all equal, namely $d = \mathrm{lcm}(a_0, \cdots, a_n)$. We define the weight vector $\mathbf{w} = (w_0, \cdots, w_n)$ and $|\mathbf{w}| = \sum_j w_j$. The link $L(a_0, \ldots, a_n)$ is the total space of an S^1 orbibundle over a projective algebraic orbifold $\mathcal{Z}$, and $e(M)$ is the orbifold Euler number of the orbibundle. Note that the Euler number of the orbifold canonical bundle of this orbifold is $\frac{d-|\mathbf{w}|}{d}$. We now consider the main case of our interest, namely, [NR78]

Lemma 3.3. *The link $L(\mathbf{a})$ is a homology 3-sphere if and only if the integers $a_0, \ldots, a_n$ are pairwise relatively prime.*

In this case the weights are $w_j = a_0 \cdots \widehat{a}_j \cdots a_n$ and the degree $d = a_0 \cdots a_n$ which satisfies $d = w_j a_j$ for all j. (This last equation holds in generally for Brieskorn manifolds). So in the case of homology spheres we have

(13) $\alpha_j = a_j, \quad s_j = 1, \quad g = 0. \quad -e(M) = \dfrac{1}{a_0 a_1 \cdots a_n} = \dfrac{1}{d}.$

The equation for β_j becomes

$$\sum_{j=0}^{n} \frac{\beta_j}{a_j} = \frac{1}{a_0 a_1 \cdots a_n}.$$

As mentioned previously every Seifert fibered manifold that is an integer homology sphere is diffeomorphic up to orientation to a Brieskorn complete intersection $L(a_0, \ldots, a_n)$ [NR78]. Furthermore, they are all homogeneous manifolds which except for the Poincaré sphere have the form $\widetilde{PSL}(2, \mathbb{R})/\Gamma$ where Γ is an infinite discrete group such that $[\Gamma, \Gamma] = \Gamma$. The Poincaré sphere is represented by $\mathbf{a} = (2, 3, 5)$. It is the homogeneous space $SU(2)/\mathbb{I}^*$ where $\mathbb{I}^*$ is the binary icosahedral group, a finite group of order 120. Conditions on the Seifert invariants that a given Seifert manifold can be represented as $\widetilde{PSL}(2, \mathbb{R})/\Gamma$ where Γ is a cocompact discrete subgroup of $\widetilde{PSL}(2, \mathbb{R})$ were given in [RV81]. Also the fact that any complete intersection link of the form (11) is a homogeneous space $\widetilde{PSL}(2, \mathbb{R})/\Gamma$ or $SU(2)/\mathbb{I}^*$ was proved in [Neu77].

Example 3.4. When $n = 2$ we have the Brieskorn 3-manifolds $L(a_0, a_1, a_2)$ which were treated extensively in [Mil75]. When $n = 3$ we can redefine the coordinates giving the complete intersection

$$z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0, \qquad c_0 z_0^{a_0} + c_1 z_1^{a_1} + c_2 z_2^{a_2} + c_3 z_3^{a_3} = 0.$$

The condition on the minor determinants is $c_i \neq c_j$ for all $i \neq j = 0, \cdots, 3$.

3.3. **Sasakian Structures on Homology Spheres.** It follows from Proposition 9.6.1 of [BG08] that the links $L(\mathbf{a})$ admit a quasiregular Sasakian structure induced from the Sasakian structure on S^{2n+1} as a complete intersection. Now $L(\mathbf{a})$ is the total space of an S^1 orbibundle over a projective algebraic variety $\mathcal{Z}_{\mathbf{w}}$ of complex dimension one, that is, a Riemann surface embedded as a complete intersection in the weighted projective space $\mathbb{CP}(\mathbf{w})$. Given the form of the weights for homology spheres, we see that $\mathbb{CP}(\mathbf{w})$ is isomorphic as a projective variety to $\mathbb{CP}^n$ and $\mathcal{Z}_{\mathbf{w}}$ is isomorphic to $\mathbb{CP}^1$ but with a non-trivial orbifold structure.

Recall from Definition 7.5.24 of [BG08] that a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is *positive (negative)* if the basic first Chern class $c_1^B(\mathcal{F}_\xi)$ can be represented by a positive (negative) definite $(1, 1)$ form.

Lemma 3.5. *Let $L(\mathbf{a})$ be a homology 3-sphere. Then except for the Poincaré sphere with link $L(2, 3, 5)$, the Sasakian structure on the link $L(\mathbf{a})$ is negative. The Sasakian structure on the Poincaré sphere $L(2, 3, 5)$ is positive.*

Proof. Note that the basic first Chern class $c_1^B(\mathcal{F}_\xi)$ is the pullback of the orbifold first Chern class which is $\frac{|\mathbf{w}| - |\mathbf{d}|}{d}$ times the area form α on $\mathbb{CP}^1$. Thus, we have

$$(14) \qquad c_1^{orb}(\mathcal{Z}_\mathbf{w}) = \frac{1}{d}\Big(\sum_j w_j - (n-1)a_0 \cdots a_n\Big)\alpha.$$

If $n = 2$ then it is well known [Mil75] and easy to see that $L(2, 3, 5)$ is the only non-standard homology sphere for which $c_1 > 0$. So we assume that $n \geq 3$ and without loss of generality we can assume that $a_0 < a_1 < \cdots < a_n$ and $a_j \geq 2$. We then have

$$
\begin{aligned}
|\mathbf{d}| - |\mathbf{w}| &= (n-1)a_0 \cdots a_n - \sum_j a_0 \cdots \widehat{a}_j \cdots a_n \\
&> (n-1)a_0 \cdots a_n - (n+1)a_1 \cdots a_n \\
&= \big((n-1)a_0 - (n+1)\big)a_1 \cdots a_n \geq (n-3)a_1 \cdots a_n \geq 0.
\end{aligned}
$$

$\square$

When M^3 is a homology sphere with infinite fundamental group arising from the link $L(\mathbf{a})$, the base $B = B(\mathbf{a})$ is S^2 with an orbifold structure consisting of $n+1$ orbifold points. So if M^3 is not the Poincaré sphere, or equivalently that $\pi_1(M^3)$ is infinite, there are $n+1$ singular orbits corresponding to setting $z_j = 0$ where $j = 0, \ldots, n$. The order of the isotropy subgroup when $z_j = 0$ is a_j, so the total order of the orbifold quotient is $d_\mathbf{a} = a_0 \cdots a_n$ which coincides with the order υ of the Sasakian structure.

We are now ready for Belgun's theorem. The version given in Theorem 10.1.3 of [BG08] is more convenient for our purposes, and we give only what we need here.

Theorem 3.6 ([Bel01]). *Let M be a 3-dimensional compact manifold admitting a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$. Then*

  (1) *If $\mathcal{S}$ is positive, M is spherical, and there is a Sasakian metric of constant Φ-sectional curvature 1 in the same deformation class as g.*

(2) *If $\mathcal{S}$ is negative, M is of $\widetilde{PSL}(2,\mathbb{R})$ type, and there is a Sasakian metric of constant Φ-sectional curvature -4 in the same deformation class as g.*

In the positive case the metric g also has constant Riemannian sectional curvature 1. Moreover, in case 1 of the theorem there is a ray of constant Φ sectional curvature c with $c > -3$ which corresponds to constant transverse holomophic sectional curvature $c + 3 > 0$. In case 2 there is a ray of constant Φ sectional curvature c with $c < -3$ which corresponds to constant transverse holomorphic sectional curvature $c + 3 < 0$. Except for the standard sphere the automorphism group of $\mathcal{S}$ is one dimensional consisting of the flow generated by the Reeb vector field. Thus, the Sasaki cone is one dimensional for these structures.

3.4. **Orbifold Kähler-Einstein Metrics on $B_\mathbf{a}$.** It follows from Theorem 3.6 and the orbifold Boothby-Wang Theorem [BG00] that $B_\mathbf{a}$ admits a Kähler-Einstein orbifold metric. For example for the Poincaré sphere with $\mathbf{a} = (2,3,5)$, the orbifold first Chern class of the base orbifold $B_\mathbf{a}$ is $c_1^{orb}(B_\mathbf{a}) = \frac{1}{30}\alpha$ where α is primitive in $H^2(B_\mathbf{a}, \mathbb{Z})$. So if $p : \mathsf{B}B_\mathbf{a} \longrightarrow B_\mathbf{a}$ is the orbifold classifying map, $p^*c_1^{orb}$ is primitive in $H^2_{orb}(B_\mathbf{a}, \mathbb{Z})$. Moreover, c_1^{orb} pulls back to an integer cohomology class on $M_\mathbf{a}^3$ which of course in our case is 0.

We want to emphasize, as mentioned previously, that the orbifold Kähler-Einstein metrics on $B_\mathbf{a}$ correspond to Sasaki-η-Einstein metrics on $M_\mathbf{a}^3$. In the positive case when $M_\mathbf{a}^3 = S^3/\mathbb{I}^*$ and $\mathbf{a} = (2,3,5)$, the positive Kähler-Einstein metric on $B_{(2,3,5)}$ with scalar curvature 8 corresponds to the standard constant sectional curvature (and Φ sectional curvature) 1 metric on $S^3/\mathbb{I}^*$. In the negative case the Kähler-Einstein metric on $B_\mathbf{a}$ with scalar curvature -2 corresponds to constant Φ sectional curvature -4 on $M_\mathbf{a}^3$. In both cases by rescaling the Kähler-Einstein metric on B_a, we obtain a ray of Sasaki-η-Einstein metrics on $M_\mathbf{a}^3$ with constant Φ sectional curvature.

In order to construct η-Einstein metrics on joins, we need to consider an *index*. In [BG00] we worked with positive Sasakian structure in which case we defined the Fano index. For complete intersections it is $|\mathbf{w}| - |\mathbf{d}|$. However, here in all but one case, the Sasakian structure is negative, so we define the *canonical index* as $I_\mathbf{a} = |\mathbf{d}| - |\mathbf{w}|$. For homology spheres $M_\mathbf{a}^3$, $|\mathbf{d}| = (n-1)a_0 \cdots a_n$ and $|\mathbf{w}| = \sum_j a_0 \cdots \hat{a}_j \cdots a_n$.

4. The Join of an Homology 3-Sphere

Here we let $M_{\mathbf{a}}^3$ be a homology sphere described by the links $L(a_0, \ldots, a_n)$ of Section 3.2 and we assume that $a_i \geq 2$ for all i, so that it is not the standard sphere. As seen in the last section $M_{\mathbf{a}}^3$ has a natural Sasakian structure of constant Φ sectional curvature, and we can easily obtain higher dimensional Sasakian manifolds with perfect fundamental group by applying the join construction.

4.1. The Topology of $M_{\mathbf{a}}^3 \star_{k,l} N$ with N Simply Connected.

Theorem 4.1. *Let N be a simply connected $2r+1$-dimensional quasi-regular Sasakian manifold of order v, and let $M_{\mathbf{a}}^3$ be an homology sphere described in Section 3.2, so $\gcd\{a_i\} = 1$. If also $\gcd(la_0 \cdots a_n, kv) = 1$ then the (k,l)-join $M_{\mathbf{a}}^3 \star_{k,l} N$ is a $2r+3$-dimensional Sasakian manifold. Moreover, if $M_{\mathbf{a}}^3$ is not the Poincaré sphere we have*

- *$M_{\mathbf{a}}^3 \star_{k,l} N$ has a perfect fundamental group isomorphic to a quotient of $\pi_1(M_{\mathbf{a}}^3)$.*
- *$\pi_1(M_{\mathbf{a}}^3 \star_{k,l} N)$ is a $\mathbb{Z}_l$ extension of the perfect group $\pi_1^{orb}(B_{\mathbf{a}}) \approx \pi_1(M_{\mathbf{a}}^3)/\mathbb{Z}$.*
- *$\pi_i(M_{\mathbf{a}}^3 \star_{k,l} N) \approx \pi_i(N)$ for $i \geq 2$.*

If $M_{\mathbf{a}}^3$ is the Poincaré sphere $S^3/\mathbb{I}^*$, then

- *If l is odd $M_{\mathbf{a}}^3 \star_{k,l} N$ has a perfect fundamental group equal to $\mathbb{I}$.*
- *If l is even $M_{\mathbf{a}}^3 \star_{k,l} N$ has a perfect fundamental group equal to either $\mathbb{I}$ or $\mathbb{I}^*$.*
- *$\pi_i(M_{\mathbf{a}}^3 \star_{k,l} N) \approx \pi_i(S^3) \oplus \pi_i(N)$ for all $i \geq 3$.*

Proof. The fact that $M_{\mathbf{a}}^3 \star_{k,l} N$ is a smooth Sasakian manifold of dimension $2r+3$ follows from Proposition 7.6.6 of [BG08]. There are two relevant fibrations involving the join. By Proposition 7.6.7 of [BG08] $M_{\mathbf{a}}^3 \star_{k,l} N$ can be realized as a $N/\mathbb{Z}_l$-bundle over the base orbifold $B_{\mathbf{a}}$ of $M_{\mathbf{a}}^3$, that is we have an orbifold fibration

$$(15) \qquad N/\mathbb{Z}_l \longrightarrow M_{\mathbf{a}}^3 \star_{k,l} N \longrightarrow B_{\mathbf{a}}$$

whose long exact homotopy sequence is

$$(16)$$
$$0 \longrightarrow \pi_2(N) \longrightarrow \pi_2(M_{\mathbf{a}}^3 \star_{k,l} N) \longrightarrow \pi_2^{orb}(B_{\mathbf{a}}) \longrightarrow \mathbb{Z}_l \longrightarrow \pi_1(M_{\mathbf{a}}^3 \star_{k,l} N) \longrightarrow \pi_1^{orb}(B_{\mathbf{a}}) \longrightarrow 1.$$

On the other hand we have the circle bundle that defines the join, namely

$$(17) \qquad S^1 \longrightarrow M_{\mathbf{a}}^3 \times N \longrightarrow M_{\mathbf{a}}^3 \star_{k,l} N,$$

which gives the long exact sequence

$$(18)$$
$$0 \longrightarrow \pi_2(M_{\mathbf{a}}^3 \times N) \longrightarrow \pi_2(M_{\mathbf{a}}^3 \star_{k,l} N) \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M_{\mathbf{a}}^3) \longrightarrow \pi_1(M_{\mathbf{a}}^3 \star_{k,l} N) \longrightarrow 1.$$

Since the quotient of a perfect group is perfect, this implies that in all cases the fundamental group of $M_{\mathbf{a}}^3 \star_{k,l} N$ is perfect.

Now if $M_{\mathbf{a}}^3$ is not the Poincaré sphere Lemma 3.2 says that for $i \geq 2$, $\pi_i^{orb}(B_{\mathbf{a}}) = 0$ which implies that $\pi_1(M_{\mathbf{a}}^3 \star_{k,l} N)$ is a $\mathbb{Z}_l$ extension of $\pi_1^{orb}(B_{\mathbf{a}})$. The second statement then follows immediately. The third statement also follows from the exact homotopy sequence and the fact that $\pi_i^{orb}(B_{\mathbf{a}}) = 0$ for $i \geq 2$.

When $M_{\mathbf{a}}^3$ is the Poincaré sphere $S^3/\mathbb{I}^*$, the long exact homotopy sequence (16) becomes, again using Lemma 3.2,
(19)
$$\longrightarrow \pi_2(N) \longrightarrow \pi_2(M_{\mathbf{a}}^3 \star_{k,l} N) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_l \longrightarrow \pi_1(M_{\mathbf{a}}^3 \star_{k,l} N) \longrightarrow \mathbb{I} \longrightarrow 1.$$

But also in this case the long exact homotopy sequence (18) gives

$$(20)\quad 0 \longrightarrow \pi_2(N) \longrightarrow \pi_2(M_{\mathbf{a}}^3 \star_{k,l} N) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{I}^* \longrightarrow \pi_1(M_{\mathbf{a}}^3 \star_{k,l} N) \longrightarrow 1.$$

From the exact sequence (20), $\pi_1(M_{\mathbf{a}}^3 \star_{k,l} N)$ is either $\mathbb{I}$ or $\mathbb{I}^*$. But from (19) if l is odd, then $\pi_1(M_{\mathbf{a}}^3 \star_{k,l} N)$ cannot be $\mathbb{I}^*$ which proves the first statement of this case. However, if l is even, $\pi_1(M_{\mathbf{a}}^3 \star_{k,l} N)$ can be either $\mathbb{I}$ or $\mathbb{I}^*$. The final statement follows from the homotopy exact sequence of (17). $\qquad\square$

Also when l is odd the map in (20)

$$\pi_2(M_{\mathbf{a}}^3 \star_{k,l} N)/\pi_2(N) \approx \mathbb{Z} \longrightarrow \mathbb{Z}$$

is multiplication by 2; whereas, the similar map in (19) is multiplication by l.

Remark 4.2. Notice that for fixed $\mathbf{a}$ and v, there are infinitely many (k, l) that satisfy the smoothness condition $\gcd(la_0 \cdots a_n, kv) = 1$.

For cohomology we have

Lemma 4.3. *Assume the hypothesis of Theorem 4.1. Then $H^2(M_{\mathbf{a}}^3 \star_{k,l} N, \mathbb{Z}) \approx H^2(N, \mathbb{Z}) \oplus \mathbb{Z}$.*

Proof. Now $M_{\mathbf{a}}^3$ is a homology sphere, and by Theorem 4.1 $M_{\mathbf{a}}^3 \star_{k,l} N$ has perfect fundamental group, so the Gysin sequence of the fibration (17) gives the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^2(M_{\mathbf{a}}^3 \star_{k,l} N, \mathbb{Z}) \longrightarrow H^2(N, \mathbb{Z}) \longrightarrow 0$$

which splits since $H^2(N, \mathbb{Z})$ is free. $\qquad\square$

A special case of interest is $N = S^{2r+1}$, the $(2r + 1)$-sphere.

Proposition 4.4. *Let $M_{\mathbf{a}}^3$ be an homology 3-sphere. Then the join $M_{\mathbf{a}}^3 \star_{k,l} S^{2r+1}$ has the rational cohomology ring of $S^2 \times S^{2r+1}$ for all relatively prime positive integers k, l, and the join $M_{\mathbf{a}}^3 \star_{k,1} S^{2r+1}$ has the*

integral cohomology ring of $S^2 \times S^{2r+1}$ for all positive integers k. If $r = 1$, the join $M^3_{\mathbf{a}} \star_{k,l} S^3$ has the integral cohomology ring of $S^2 \times S^3$ for all relatively prime positive integers k, l.

Proof. Our proof uses the spectral sequence method employed in [WZ90, BG00] (see also Section 7.6.2 of [BG08]). The fibration (17) together with the torus bundle with total space $M^3_{\mathbf{a}} \times S^{2r+1}$ gives the commutative diagram of fibrations

(21)

$$
\begin{array}{ccccc}
M^3_{\mathbf{a}} \times S^{2r+1} & \longrightarrow & M^3_{\mathbf{a}} \star_{k,1} S^{2r+1} & \longrightarrow & \mathsf{B}S^1 \\
\downarrow = & & \downarrow & & \downarrow \psi \\
M^3_{\mathbf{a}} \times S^{2r+1} & \longrightarrow & \mathsf{B}B_{\mathbf{a}} \times \mathbb{CP}^r & \longrightarrow & \mathsf{B}S^1 \times \mathsf{B}S^1
\end{array}
$$

where $\mathsf{B}G$ is the classifying space of a group G or Haefliger's classifying space [Hae84] of an orbifold if G is an orbifold. Note that by definition $H^i_{orb}(B_{\mathbf{a}}, \mathbb{Z}) = H^i(\mathsf{B}B_{\mathbf{a}}, \mathbb{Z})$ and an easy argument of the orbifold fibration $S^1 \rightarrow M^3_{\mathbf{a}} \rightarrow B_{\mathbf{a}}$ shows that $H^*_{orb}(B_{\mathbf{a}}, \mathbb{Z}) = H^*(S^2, \mathbb{Z})$. So cohomologically, $\mathsf{B}B_{\mathbf{a}}$ is S^2. We also note that $\mathsf{B}S^1 = \mathbb{CP}^\infty$ with cohomology ring $\mathbb{Z}[s]$ with $s \in H^2(\mathbb{CP}^\infty, \mathbb{Z})$.

Now the map ψ is that induced by the inclusion $e^{i\theta} \mapsto (e^{il\theta}, e^{-ik\theta})$. So writing

$$H^*(\mathsf{B}S^1 \times \mathsf{B}S^1, \mathbb{Z}) = \mathbb{Z}[s_1, s_2]$$

we see that $\psi^* s_1 = ls$ and $\psi^* s_2 = -ks$. The E_2 term of the Leray-Serre spectral sequence of the top fibration of diagram (21) is $E_2^{p,q} = H^p(\mathsf{B}S^1, H^q(M^3_{\mathbf{a}} \times S^{2r+1}, \mathbb{Z}))$. The non-zero terms occur when p is even, say $2p'$, and $q = 0, 3, 2r+1, 2r+4$. Let α, β denote the orientation class of $M^3_{\mathbf{a}}, S^{2r+1}$, respectively. The cohomology ring of the fiber is $\Lambda[\alpha, \beta]$, whereas, that of the base is $\mathbb{Z}[s]$. Then if $r > 1$ the differential $d_4(\alpha) = (ls)^2$, so we need $l = 1$ to avoid torsion in $H^4 \approx$ torsion in H_3. Then by naturality we have $d_4(\alpha \otimes s^{2p'}) = s^{2p'+2}$, but then $d_{2r+2}(\beta) = 0$ and $d_{2r+2}(\alpha \cup \beta) = s^2 \otimes \beta$, so the classes that survive are $s, \beta, \beta \cup s$ which proves the result for $r > 1$.

When $r = 1$ we have $d_4(\alpha) = l^2 s^2$ and $d_4(\beta) = k^2 s^2$ which implies that the primitive integral class $k^2\alpha - l^2\beta \in E_2^{0,3}$ survives, and this proves the result together with Poincaré duality. $\square$

4.2. **The Topology of $M^3_{\mathbf{a}} \star_{k,l} M^3_{\mathbf{b}}$.** Here we consider the join of two homology 3-spheres. We assume that $\gcd(d_{\mathbf{a}}, d_{\mathbf{b}}) = 1$ which implies that at most one of them can be the Poincaré sphere. From the fibration (17) with N replaced by $M^3_{\mathbf{b}}$ we obtain

$$0 \longrightarrow \pi_2(M^3_{\mathbf{a}} \times M^3_{\mathbf{b}}) \longrightarrow \pi_2(M^3_{\mathbf{a}} \star_{k,l} M^3_{\mathbf{b}}) \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M^3_{\mathbf{a}} \times M^3_{\mathbf{b}}) \longrightarrow \pi_1(M^3_{\mathbf{a}} \star_{k,l} M^3_{\mathbf{b}}) \longrightarrow 1.$$

Now $M_{\mathbf{a}}^3 \times M_{\mathbf{b}}^3$ has perfect fundamental group $\pi_1(M_{\mathbf{a}}^3) \times \pi_1(M_{\mathbf{b}}^3)$, and if neither is the Poincaré sphere $M_{\mathbf{a}}^3 \times M_{\mathbf{b}}^3$ is aspherical. Hence, $\pi_1(M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3)$ is perfect and $\pi_i(M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3) = 0$ for $i \geq 3$. On the other hand, from the orbibundle $S^1 \longrightarrow M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3 \longrightarrow B_{\mathbf{a}} \times B_{\mathbf{b}}$ we have

(22)
$$0 \longrightarrow \pi_2(M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3) \longrightarrow \pi_2^{orb}(B_{\mathbf{a}}) \times \pi_2^{orb}(B_{\mathbf{b}}) \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3) \longrightarrow \pi_1^{orb}(B_{\mathbf{a}}) \times \pi_1^{orb}(B_{\mathbf{b}}) \longrightarrow 1.$$

Now $\pi_1^{orb}(B_{\mathbf{a}})$ and $\pi_1^{orb}(B_{\mathbf{b}})$ vanish by Lemma 3.2, so $\pi_2(M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3) = 0$ making $M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3$ aspherical and its fundamental group is perfect, but not simple.

Concerning the cohomology of $M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3$ we have:

Proposition 4.5. *For all relatively prime positive integers k, l the 5-manifold $M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3$ has the integral cohomology ring of $S^2 \times S^3$.*

Proof. The proof is similar to the $r = 1$ case of Proposition 4.4, so the details are left to the reader. □

Combining this theorem with the above analysis gives

Theorem 4.6. *The 5-manifolds $M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3$ are aspherical with perfect fundamental group and the integral cohomology ring of $S^2 \times S^3$.*

4.3. **Distinguishing Contact Structures.** The crudest invariant of a contact structure is the first Chern class $c_1(\mathcal{D})$ of the contact bundle; nevertheless, it can distinguish countably many contact structures in many cases. A much more subtle contact invariant is the contact homology of Eliashberg, Giventhal, and Hofer [EGH00] employed for example in [BP12]. However, in the present paper we cannot even pin down the diffeomorphism type, so we only make use of c_1 to distinguish contact structures.

Here we consider only a special case where given two base orbifolds B_1 and B_2, we assume that $p^* c_1^{orb}(B_i) = -I_i \alpha_i$ where I_i is the canonical index[1] of B_i and $\alpha_i \in H_{orb}^2(B_i, \mathbb{Z})$ is a generator. In this case we have

(23)
$$p^* c_1^{orb}(B_1 \times B_2) = -I_1 \alpha_1 - I_2 \alpha_2.$$

Now consider the S^1 bundle determined by the Kähler class $k_1 \alpha_1 + k_2 \alpha_2$ on $B_1 \times B_2$. Let $\pi : M_1 \star_{k_1,k_2} M_2 \longrightarrow B_1 \times B_2$ denote the bundle projection. By the join construction we know that $\pi^*(k_1 \alpha_1 + k_2 \alpha_2) = [d\eta] = 0$. So there is a generator $\gamma \in H^2(M_1 \star_{k_1,k_2} M_2, \mathbb{Z})$ such that $\pi^* \alpha_1 = k_2 \gamma$ and $\pi^* \alpha_2 = -k_1 \gamma$. Now the first Chern class of the contact

---

[1]Since we mainly deal with negative Sasakian structures, we use the canonical index instead of the Fano index used in [BG00]. Of course, the Fano index is just the negative of the canonical index.

bundle is the pullback of the orbifold first Chern class on $B_1 \times B_2$, that is,

$$(24) \quad c_1(\mathcal{D}) = \pi^* c_1^{orb}(B_1 \times B_2) = -I_1 \pi^* \alpha_1 - I_2 \pi^* \alpha_2 = (I_2 k_1 - I_1 k_2)\gamma.$$

The mod 2 reduction of $c_1(\mathcal{D})$ is a topological invariant, namely the second Stiefel-Whitney class $w_2(M_1 \star_{k_1,k_2} M_2) \in H^2(M_1 \star_{k_1,k_2} M_2, \mathbb{Z}_2)$. This allows us to distinguish different manifolds with the same perfect fundamental group.

5. Extremal Sasaki Metrics on Joins of Homology Spheres

We shall always assume that $M_{\mathbf{a}}^3$ is an homology 3-sphere with constant Φ sectional curvature either 1 or -4 and that it is not the standard sphere. So it is either the Poincaré sphere with constant sectional curvature 1 or a negative homology sphere with constant Φ sectional curvature -4.

Theorem 5.1. *Let N be a simply connected quasi-regular Sasakian manifold of dimension $2r + 1$ which fibers in the orbifold sense over a projective algebraic orbifold $\mathcal{Z}$ with an orbifold Kähler metric h of constant scalar curvature S. Let υ be the order of the Sasakian structure on N and assume that $\gcd(la_0 \cdots a_n, k\upsilon) = 1$. Then the join $M_{\mathbf{a}}^3 \star_{k,l} N$ is a smooth Sasakian manifold of dimension $2r + 3$ with perfect fundamental group and the induced Sasakian structure has a ray of extremal CSC Sasaki metrics.*

Proof. The proof easily follows from the join construction together with Theorem 4.1. □

When N has a Sasakian structure whose automorphism group $\mathfrak{Aut}(\mathcal{S})$ has dimension greater than 1, one can deform in the Sasaki cone to obtain new extremal Sasakian structures. Indeed, the Openness Theorem of [BGS08] guarentees the existence of an open set of such extremal Sasaki metrics.

5.1. **Extremal Sasakian metrics on** $M_{\mathbf{a}}^3 \star_{k,l} S^{2r+1}$**.** We can easily obtain constant scalar curvature Sasakian metrics on manifolds $M_{\mathbf{a}}^3 \star_{k,l} S^{2r+1}$ from the lift of the product Kähler orbifold metric on $B(\mathbf{a}) \times \mathbb{CP}^r$. Let us describe the orbifold structure on $B(\mathbf{a})$. As an algebraic variety it is $\mathbb{CP}^1$ with $n + 1$ distinct marked points. Thus, as an algebraic variety the product $B(\mathbf{a}) \times \mathbb{CP}^r$ is $\mathbb{CP}^1 \times \mathbb{CP}^r$ with a non-trivial orbifold structure on the first factor described by branch divisors $\Delta = \sum_i \Delta_i$. On the first factor we have an orbifold Kähler-Einstein metric with scalar curvature -2, and on the second factor the

standard Fubini-Study metric with constant scalar curvature $4r(r+1)$. Since there are an infinite number of integers $l, k, a_0, \ldots, a_n$ that satisfy $\gcd(la_0 \cdots a_n, k) = 1$, Theorems 4.1 and 5.1, and the results of Section 4.3 imply

Theorem 5.2. *Let $M_{\mathbf{a}}^3$ be a negative homology sphere and assume that $\gcd(la_0 \cdots a_n, k) = 1$. Then there is a countably infinite number of $(2r + 3)$-dimensional contact manifolds $M_{\mathbf{a}}^3 \star_{k,l} S^{2r+1}$ with a perfect fundamental group that admit a ray of extremal CSC Sasaki metrics.*

Then Proposition 4.4 and Theorem 5.2 immediately give Theorem 1.1 of the introduction.

5.2. **Extremal Sasakian metrics on $M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3$.** Notice that since both homology 3-spheres have constant Φ-sectional curvature, the bases have constant holomorphic sectional curve; hence, they both have CSC Kähler orbifold metrics. So if $\gcd(la_0 \cdots a_n, kb_0 \cdots b_n) = 1$, the 5-manifold $M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3$ has a CSC Sasaki metric. This includes the case when one of the homology 3-spheres is the Poincaré sphere.

When both homology 3-spheres are negative, we obtain Sasaki-η-Einstein metrics in certain cases. In order to get such metrics we need to select the S^1 orbibundle over $B_{\mathbf{a}} \times B_{\mathbf{b}}$ whose Euler class is proportional to the orbifold first Chern class $c_1^{orb}(B_{\mathbf{a}} \times B_{\mathbf{b}})$ with negative proportionality constant. From Equation (14) we have with obvious notation

$$(25) \qquad c_1^{orb}(B_{\mathbf{a}} \times B_{\mathbf{b}}) = \frac{|\mathbf{w_a}| - |\mathbf{d_a}|}{d_{\mathbf{a}}}\alpha + \frac{|\mathbf{w_b}| - |\mathbf{d_b}|}{d_{\mathbf{b}}}\beta.$$

So by Equation (24) to obtain a Sasaki-η-Einstein metric on $M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3$ we need $c_1(\mathcal{D}) = \pi^* c_1^{orb}(B_{\mathbf{a}} \times B_{\mathbf{b}}) = 0$, and we do this by choosing $k = |\mathbf{d_a}| - |\mathbf{w_a}| = I_{\mathbf{a}}$ and $l = |\mathbf{d_b}| - |\mathbf{w_b}| = I_{\mathbf{b}}$ as long as they are relatively prime. As in [BG00] to handle the case when they are not relatively prime, we define the *relative indices* by

$$\mathcal{I}_{\mathbf{a}} = \frac{I_{\mathbf{a}}}{\gcd(I_{\mathbf{a}}, I_{\mathbf{b}})}, \qquad \mathcal{I}_{\mathbf{b}} = \frac{I_{\mathbf{b}}}{\gcd(I_{\mathbf{a}}, I_{\mathbf{b}})}.$$

Then $\gcd(\mathcal{I}_{\mathbf{a}}, \mathcal{I}_{\mathbf{b}}) = 1$. So generally we obtain a Sasaki-η-Einstein metric on $M_{\mathbf{a}}^3 \star_{k,l} M_{\mathbf{b}}^3$ by choosing $k = \mathcal{I}_{\mathbf{a}}$ and $l = \mathcal{I}_{\mathbf{b}}$. We also make note of the easily shown fact that $\gcd(d_{\mathbf{a}}, I_{\mathbf{a}}) = 1$ for all $\mathbf{a}$ with the a_i pairwise relatively prime. This guarentees that as long as $\gcd(d_{\mathbf{a}}, d_{\mathbf{b}}) = 1$, there is a Sasaki-$\eta$-Einstein metric on $M_{\mathbf{a}}^3 \star_{\mathcal{I}_{\mathbf{a}}, \mathcal{I}_{\mathbf{b}}} M_{\mathbf{b}}^3$. Summarizing we have

Theorem 5.3. *Let $M_{\mathbf{a}}^3$ and $M_{\mathbf{b}}^3$ be negative homology 3-spheres with canonical indices $I_{\mathbf{a}}$ and $I_{\mathbf{b}}$, respectively. Then*

(1) If $\gcd(ld_\mathbf{a}, kd_\mathbf{b}) = 1$ *the 5-dimensional contact manifold* $(M_\mathbf{a}^3 \star_{k,l} M_\mathbf{b}^3, \mathcal{D})$ *with* $c_1(\mathcal{D}) = (kI_\mathbf{b} - lI_\mathbf{a})\gamma$ *admits a CSC Sasaki metric.*

(2) If $\gcd(d_\mathbf{a}, d_\mathbf{b}) = 1$, *the 5-dimensional contact manifold* $(M_\mathbf{a}^3 \star_{\mathcal{J}_\mathbf{a}, \mathcal{J}_\mathbf{b}} M_\mathbf{b}^3, \mathcal{D})$ *with* $c_1(\mathcal{D}) = 0$ *admits a ray of negative Sasaki-η-Einstein metrics; hence, it also admits Lorentzian Sasaki-Einstein metrics.*

Since for fixed $\mathbf{a}$ and $\mathbf{b}$ there are a countably infinite number of relatively prime pairs (k, l) that satisfy $\gcd(ld_\mathbf{a}, kd_\mathbf{b}) = 1$, there are a countably infinite number of such contact 5-manifolds.

Combining Theorems 4.6 and 5.3 proves Theorem 1.2 of the Introduction.

5.3. **More Sasaki-η-Einstein Manifolds.** It is now straightforward to construct higher dimensional examples of η-Einstein and Lorentzian Sasaki-Einstein metrics. Since as mentioned above $\gcd(d_\mathbf{a}, I_a) = 1 = \gcd(\mathcal{J}_\mathbf{a}, \mathcal{J})$, we have

Theorem 5.4. *Let $M_\mathbf{a}^3$ be a negative homology 3-sphere and let N be a simply connected negative Sasakian manifold with canonical index I and order υ. Let $\mathcal{J}_\mathbf{a}, \mathcal{J}$ denote the relative canonical indices. Then if $\gcd(d_\mathbf{a}\mathcal{J}, \upsilon) = 1$ the manifold $M_\mathbf{a}^3 \star_{\mathcal{J}_\mathbf{a}, \mathcal{J}} N$ has perfect fundamental group and admits a ray of negative Sasaki-η-Einstein metrics, and hence, a Lorentzian Sasaki-Einstein metric.*

Proof of Theorem 1.3. In order to prove Theorem 1.3 we need to find simply connected negative eta-Einstein 5-manifolds with arbitrary second Betti number b_2 that satisfy the hypothesis of Theorem 5.4. Then by Lemma 4.3 we will have arbitrary $b_2(M_\mathbf{a}^3 \star_{\mathcal{J}_\mathbf{a}, \mathcal{J}} N) = b_2(N) + 1 \geq 1$. In order to find such N we make use of the work of Gomez [Gom11] on negative η-Einstein structures on arbitrary connected sums of $S^2 \times S^3$. To exhaust all connected sums $l(S^2 \times S^3)$ we divide the analysis into seven cases, two infinite series and five sporatic cases. In each case we need only to check that $\gcd(I, \upsilon) = 1$. Then we can always find $\mathbf{a}$ such $\gcd(d_\mathbf{a}, \upsilon) = 1$. The first series is given by the link of the hypersurface $z_0^4 + z_1^{8k+2} + z_2^{4k+1}z_3 + z_3^{2k+1}z_2 = 0$. Here $k \geq 1$ and the second Betti number b_2 is $l = 2k + 1$. One checks that this has $\upsilon = 2(4k + 1)$ and $I = 16k(4k + 1) - (4k + 3)^2$. It follows that $\gcd(I, \upsilon) = 1$ in this case. The second series applies to $b_2 = k - 1$ with $k \geq 9$ and odd. Here the hypersurface is $z_0^4 + z_1^2 + z_2^k + z_3^k = 0$ which has $I = k - 8$ and $\upsilon = 2k$. One easily sees that $\gcd(I, \upsilon) = 1$ in this case. The remaining five cases are $l = 0, 1, 2, 4, 6$. The last two are represented in [Gom11]; however, their weights and indices are quite large, but more importantly, the $l = 4$ case has $\gcd(I, \upsilon) = 11$, so the join will not be smooth. The

case $l = 6$ in [Gom11] does satisfy this condition, so it can be used. Nevertheless, in the table below we give simpler polynomials to cover these five sporatic cases. We used Orlik's formula (see Corollary 9.3.13 in [BG08]) with a Maple program to compute the second Betti number b_2. Note that they are not necessarily connected sums of $S^2 \times S^3$, as there may be torsion in $H_2(N, \mathbb{Z})$.

Sporatic Cases with $I = 1$

b_2	$\mathbf{w}$	Polynomial
0	(5,6,6,8)	$z_0^8 + z_1^4 + z_2^4 + z_3^3$
1	(2,4,6,11)	$z_0^{12} + z_1^6 + z_2^4 + z_3^2 z_0$
2	(6,7,28,42)	$z_0^{14} + z_1^{12} + z_2^3 + z_3^2$
4	(4,5,20,30)	$z_0^{15} + z_1^{12} + z_2^3 + z_3^2$
6	(3,4,12,16)	$z_0^{12} + z_1^9 + z_2^3 + z_3^2 z_1$

$\square$

5.4. **Sasaki-Einstein Metrics on Manifolds with Perfect Fundamental Group.** Since the only positive homology 3-sphere with perfect fundamental group is the Poincaré sphere $S^3/\mathbb{I}^*$, only joins with $S^3/\mathbb{I}^*$ can give Sasaki-Einstein metrics. Furthermore, since from Belgun's Theorem 3.6 the positive case corresponds to the bi-invariant constant sectional curvature 1 metric, these can all be obtained as quotients by $\mathbb{I}^*$ of the join of the standard S^3 with any simply connected Sasaki-Einstein manifold. So we can consider all the examples in [BG00] which involve a join of S^3 with a simply connected Sasaki-Einstein manifold. We only need to choose k, l to obtain a monotone S^1 orbibundle, that is, its cohomology class is primitive and proportional to the first Chern class of the orbifold anti-canonical bundle $K_{orb}^{-1}(\mathcal{Z})$ with $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$. We also need to scale the orbifold Kähler metric h on $\mathcal{Z}$ so that its scalar curvature is $4(n_1 + n_2)(n_1 + n_2 + 1)$ where the orbifold $\mathcal{Z}_i$ has complex dimension n_i.

The Poincaré sphere can be represented by the link of the polynomial $z_0^5 + z_1^3 + z_2^2$ with weight vector $\mathbf{w} = (6, 10, 15)$ and degree $d = 30$. It has Fano index $I_F = |\mathbf{w}| - d = 31 - 30 = 1$. So using Theorems 4.1 and 4.4 we obtain Theorem 1.4 of the Introduction.

REFERENCES

[Ban97]   A. Banyaga, *The structure of classical diffeomorphism groups*, Mathematics and its Applications, vol. 400, Kluwer Academic Publishers Group, Dordrecht, 1997. MR 1445290 (98h:22024)

[Bel01]   F. A. Belgun, *Normal CR structures on compact 3-manifolds*, Math. Z. **238** (2001), no. 3, 441–460. MR 2002k:32065

[BG00]   C. P. Boyer and K. Galicki, *On Sasakian-Einstein geometry*, Internat. J. Math. **11** (2000), no. 7, 873–909. MR 2001k:53081

[BG08]   Charles P. Boyer and Krzysztof Galicki, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008. MR MR2382957 (2009c:53058)

[BGK05] C. P. Boyer, K. Galicki, and J. Kollár, *Einstein metrics on spheres*, Ann. of Math. (2) **162** (2005), no. 1, 557–580. MR 2178969 (2006j:53058)

[BGM06] C. P. Boyer, K. Galicki, and P. Matzeu, *On eta-Einstein Sasakian geometry*, Comm. Math. Phys. **262** (2006), no. 1, 177–208. MR 2200887 (2007b:53090)

[BGO07] Charles P. Boyer, Krzysztof Galicki, and Liviu Ornea, *Constructions in Sasakian geometry*, Math. Z. **257** (2007), no. 4, 907–924. MR MR2342558 (2008m:53103)

[BGS08] Charles P. Boyer, Krzysztof Galicki, and Santiago R. Simanca, *Canonical Sasakian metrics*, Commun. Math. Phys. **279** (2008), no. 3, 705–733. MR MR2386725

[Bla10] David E. Blair, *Riemannian geometry of contact and symplectic manifolds*, second ed., Progress in Mathematics, vol. 203, Birkhäuser Boston Inc., Boston, MA, 2010. MR 2682326

[BN10]   C. P. Boyer and M. Nakamaye, *On Sasaki-Einstein manifolds in dimension five*, Geom. Dedicata **144** (2010), 141–156.

[Boy10] Charles P. Boyer, *Maximal tori in contactomorphism groups*, preprint; arXiv:math.SG/1003.1903 (2010).

[Boy11] _____, *Extremal Sasakian metrics on S^3-bundles over S^2*, Math. Res. Lett. **18** (2011), no. 1, 181–189. MR 2756009 (2012d:53132)

[BP12]   Charles P. Boyer and Justin Pati, *On the equivalence problem for toric contact structures on S^3-bundles over S^2*, math SG1204.2209 (2012).

[BTF11] Charles P. Boyer and Christina W. Tønnesen-Friedman, *Extremal Sasakian geometry on $T^2 \times S^3$ and cyclic quotients*, preprint; arXiv:math.DG/1108.2005 (2011).

[BTF12] _____, *Extremal Sasakian geometry on S^3-bundles over Riemann surfaces*, to appear (2012).

[Che11] Xiaoyang Chen, *On the fundamental groups of compact Sasakian manifolds*, arXiv:1110.2668 math.GT (2011).

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal. (2000), no. Special Volume, Part II, 560–673, GAFA 2000 (Tel Aviv, 1999). MR MR1826267 (2002e:53136)

[Gom11] Ralph R. Gomez, *Lorentzian Sasaki-Einstein metrics on connected sums of $S^2 \times S^3$*, Geom. Dedicata **150** (2011), 249–255. MR 2753706 (2012d:53139)

[Hae84] A. Haefliger, *Groupoïdes d'holonomie et classifiants*, Astérisque (1984), no. 116, 70–97, Transversal structure of foliations (Toulouse, 1982). MR 86c:57026a

[Kol05] J. Kollár, *Einstein metrics on five-dimensional Seifert bundles*, J. Geom. Anal. **15** (2005), no. 3, 445–476. MR 2190241

[Kol07] J. Kollár, *Einstein metrics on connected sums of $S^2 \times S^3$*, J. Differential Geom. **77** (2007), no. 2, 259–272. MR 2269781

[Loo84] E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Mathematical Society Lecture Note Series, vol. 77, Cambridge University Press, Cambridge, 1984. MR 747303 (86a:32021)

[LR10] Kyung Bai Lee and Frank Raymond, *Seifert fiberings*, Mathematical Surveys and Monographs, vol. 166, American Mathematical Society, Providence, RI, 2010. MR 2731766

[Mil75] J. Milnor, *On the 3-dimensional Brieskorn manifolds $M(p, q, r)$*, Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox), Princeton Univ. Press, Princeton, N. J., 1975, pp. 175–225. Ann. of Math. Studies, No. 84. MR 54 #6169

[Neu77] Walter D. Neumann, *Brieskorn complete intersections and automorphic forms*, Invent. Math. **42** (1977), 285–293. MR 0463493 (57 #3442)

[NR78] Walter D. Neumann and Frank Raymond, *Seifert manifolds, plumbing, μ-invariant and orientation reversing maps*, Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977), Lecture Notes in Math., vol. 664, Springer, Berlin, 1978, pp. 163–196. MR 518415 (80e:57008)

[Ran75] R. C. Randell, *The homology of generalized Brieskorn manifolds*, Topology **14** (1975), no. 4, 347–355. MR 54 #1270

[Ruk95] P. Rukimbira, *Chern-Hamilton's conjecture and K-contactness*, Houston J. Math. **21** (1995), no. 4, 709–718. MR 96m:53032

[RV81] Frank Raymond and Alphonse T. Vasquez, *3-manifolds whose universal coverings are Lie groups*, Topology Appl. **12** (1981), no. 2, 161–179. MR 612013 (82i:57011)

[Sav02] N. Saveliev, *Invariants for homology 3-spheres*, Encyclopaedia of Mathematical Sciences, vol. 140, Springer-Verlag, Berlin, 2002, Low-Dimensional Topology, 1. MR 2004c:57026

[Sei33] H. Seifert, *Topologie Dreidimensionaler Gefaserter Räume*, Acta Math. **60** (1933), no. 1, 147–238. MR 1555366

[ST80] Herbert Seifert and William Threlfall, *Seifert and Threlfall: a textbook of topology*, Pure and Applied Mathematics, vol. 89, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Translated from the German edition of 1934 by Michael A. Goldman, With a preface by Joan S. Birman, With "Topology of 3-dimensional fibered spaces" by Seifert, Translated from the German by Wolfgang Heil. MR 575168 (82b:55001)

[Wol67] Joseph A. Wolf, *Spaces of constant curvature*, McGraw-Hill Book Co., New York, 1967. MR 0217740 (36 #829)

[WZ90] M. Y. Wang and W. Ziller, *Einstein metrics on principal torus bundles*, J. Differential Geom. **31** (1990), no. 1, 215–248. MR 91f:53041

[Zim11] Bruno Zimmermann, *On finite groups acting on spheres and finite subgroups of orthogonal groups*, arXiv:1108.2602v1 math.GT (2011).

Charles P. Boyer, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131.

E-mail addresses: `cboyer@math.unm.edu`

CHRISTINA W. TØNNESEN-FRIEDMAN, DEPARTMENT OF MATHEMATICS, UNION COLLEGE, SCHENECTADY, NY 12308, USA

E-mail addresses: `tonnesec@union.edu`