

ON SASAKI-EINSTEIN MANIFOLDS IN DIMENSION FIVE

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ABSTRACT. We prove the existence of Sasaki-Einstein metrics on certain simply connected 5-manifolds where until now existence was unknown. All of these manifolds have non-trivial torsion classes. On several of these we show that there are a countable infinity of deformation classes of Sasaki-Einstein structures.

1. INTRODUCTION

Sasakian geometry has recently proven to be a rich source for the production of positive Einstein metrics. In [BG01] a method for obtaining such Sasaki-Einstein metrics was introduced and developed in much more detail in [BGN02a, BGN02b, BGN03b, BG03, BGK05, BGKT05, BG06, BG05]. In particular, dimension 5 is quite tractable due the Smale-Barden [Sma62, Bar65] classification of simply connected 5-manifolds. However, it is well-known that any simply connected Sasaki-Einstein manifold must be spin, so Smale's classification of simply connected spin 5-manifolds is what is needed for us. We refer to such manifolds as *Smale manifolds*. It is convenient to divide the Smale manifolds into 3 classes: the simply connected rational homology spheres, including S^5 ; the torsion free Smale manifolds with positive second Betti number, that is the k -fold connected sums of $S^2 \times S^3$ with $k \geq 1$; and the mixed type, that is, connected sums of torsion free Smale manifolds with simply connected rational homology spheres.

An important question in Riemannian geometry involves varying notions of positive curvature. In order of decreasing strength they are: positive sectional curvature, positive Ricci curvature, and positive scalar curvature. In particular, it is of much interest to determine which manifolds admit or cannot admit Riemannian metrics of positive Ricci curvature. It follows from the transverse version of the Calabi-Yau theorem [EKA90, BGN03a, BGN03c] that positive Sasaki-Seifert structures give Sasakian metrics of positive Ricci curvature. Of course, since any Sasaki-Einstein metric has positive Einstein constant, they automatically give Riemannian metrics of positive Ricci curvature. We are, therefore, interested in the following:

Problems:

- (1) Determine precisely which Smale manifolds admit positive Sasakian structures.
- (2) Determine the cardinality of the set of positive Sasaki-Seifert structures on each Smale manifold.
- (3) Determine which Smale manifolds admit Sasaki-Einstein metrics.

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- (4) Determine the cardinality of the set of deformation classes of Sasaki-Einstein structures.

Note that (3) is a subset of (1) and (4) of (2). All four problems have been completely solved for $k(S^2 \times S^3)$ [BGN03a, BGN03b, Kol07]. In the case of simply connected rational homology spheres, problem (1) has been solved completely [Kol05], problem (2) is partially solved, and problem (3) is done with the exception of the manifolds nM_2 for $n \geq 2$ [BG06, Kol05, BG08]. Moreover, nM_2 admits infinitely many positive Sasaki-Seifert structures for each $n > 0$ which can be realized by links of weighted homogeneous polynomials (see Theorem 10.2.9 of [BG08]). Recall that Kollár [Kol05] has shown that in order for a simply connected 5-manifold to admit a positive Sasakian structure very stringent conditions on the torsion subgroup of $H_2(M, \mathbb{Z})$ must hold. Specifically

Theorem 1.1 (Kollár). *Let M be a simply connected 5-manifold and suppose that M admits a positive Sasakian structure. Then $H_2(M, \mathbb{Z})_{\text{tor}}$ must be one of the following groups:*

$$\mathbb{Z}_m^2, \mathbb{Z}_2^{2n}, \mathbb{Z}_3^4, \mathbb{Z}_3^6, \mathbb{Z}_3^8, \mathbb{Z}_4^4, \mathbb{Z}_5^4 \text{ with } n > 1 \text{ and } m \geq 1.$$

In particular, if M admits a Sasaki-Einstein metric, then $H_2(M, \mathbb{Z})_{\text{tor}}$ must be one of these groups. (Note that $m = 1$ means zero torsion.)

Until now what is known about the existence of Sasaki-Einstein metrics as well as positive Ricci curvature Sasakian metrics on Smale manifolds appears in Chapters 10 and 11 of [BG08]. In fact, Corollary 11.4.14 of [BG08] gives a list of all Smale manifolds that can possibly admit a Sasaki-Einstein metric, and indicates those for which existence is known. It is the purpose of this note to present some new results on the existence of Sasaki-Einstein metrics on simply connected 5-manifolds. In particular, we show that the answer to parts (iv) and (v) of Question 11.4.1 of [BG08] is in the affirmative, but so far we have been unable to determine all the manifolds that can occur. Specifically, we prove

Theorem A. *We have:*

- (1) *The rational homology spheres*

$$2M_2, \quad 3M_2, \quad 5M_2, \quad 6M_2, \quad 7M_2$$

admit Sasaki-Einstein metrics.

- (2) *There exist Sasaki-Einstein metrics on the mixed Smale manifolds*

$$M_\infty \# M_3, M_\infty \# M_4, M_\infty \# M_5, M_\infty \# M_6, M_\infty \# M_7, M_\infty \# 2M_3, M_\infty \# 3M_3, \\ 2M_\infty \# M_2, 2M_\infty \# M_3, 2M_\infty \# M_4, 2M_\infty \# M_5, 3M_\infty \# M_2, 3M_\infty \# M_3, 3M_\infty \# M_4, \\ 4M_\infty \# M_2, 4M_\infty \# M_3, 4M_\infty \# 2M_2, 5M_\infty \# M_2, 5M_\infty \# 2M_2, 6M_\infty \# M_2.$$

- (3) *Moreover, the manifolds $2M_3, M_\infty \# M_3, M_\infty \# 2M_3$ and $3M_\infty \# M_3$ admit a countable infinity of Sasaki-Seifert structures each having Sasaki-Einstein metrics.*

Compared with our earlier work [BGN02a, BGN02b, BGN03b, BG03, BGK05, BGKT05, BG06], the proof of this theorem requires two new ingredients, first, Kollár's Theorem 1.1 [Kol05], and second, a generalization of previous estimates used in [BGN03b] to the case when branch divisors are present. The latter is Lemma 2.3 below.

The manifolds in part (3) of Theorem A are the first examples of an infinite number of deformation classes of Sasaki-Einstein metrics on Smale manifolds with torsion. The question as to which Smale manifolds admit positive Sasaki-Seifert structures that can be represented by links of isolated hypersurface singularities by weighted homogeneous polynomials is in itself an interesting one. Indeed, looking at the table in Theorem 10.2.25 of [BG08] we see that all Smale manifolds of the form $kM_\infty \# M_m$ with $k = 1, \dots, 8$, and $m \geq 12$ admit positive Sasaki-Seifert structures represented by isolated hypersurface singularities by weighted homogeneous polynomials except $5M_\infty \# M_m$. Why this happens is not understood. Furthermore, on the manifolds $6M_\infty \# M_m, 7M_\infty \# M_m, 8M_\infty \# M_m$ with $m \geq 12$ there is a unique positive Sasaki-Seifert structure and it can be represented by such a link. Generally, it seems to be not very well understood exactly why and when positive Sasaki-Seifert structures can be represented by links of isolated hypersurface singularities by weighted homogeneous polynomials. It seems quite remarkable that most of the known positive Sasaki-Seifert structures can be represented by such links. Keep in mind, however, that the complete resolution of problem (1) above is still open, and that the present paper treats only 10 of the 19 normal forms obtained by Yau and Yu [YY05] for links of weighted homogeneous polynomials, so exactly how effective our method is, remains to be seen. Furthermore, there is a generalization of our methods to links of complete intersections of weighted homogeneous polynomials where almost nothing is known.

For some reason there are gaps appearing in the types of manifolds where our method is successful. For example, $4M_2$ is proving elusive as far as the existence of Sasaki-Einstein metrics. We have found several infinite series of positive Sasaki-Seifert structures on $4M_2$, including the double infinite series $\mathbf{w} = (2l, 2(2k+1), (9l-1)(2k+1), 9l(2k+1))$ of degree $d = 18l(2k+1)$ with $\gcd(l, 2k+1) = 1$. The index is $I = 2l + 2k + 1$. No Sasaki-Einstein metrics are known to exist on these links, and for $2k > 4l - 1$ or $l > 5k + 2$ no Sasaki-Einstein metrics can exist by the Lichnerowicz obstruction described in [GMSY07]. Again the exact reason seems elusive.

Theorem B. *In addition to the Smale manifolds described in Section 10.2.1 of [BG08] and those of Theorem A, the following Smale manifolds admit a countable infinity of positive Sasaki-Seifert structures; hence, each such Sasaki-Seifert structure has Sasakian metrics of positive Ricci curvature:*

$$M_4, M_6, M_\infty \# M_4, 2M_\infty \# 4M_2, 2M_\infty \# M_3, 3M_\infty \# M_2, 3M_\infty \# 3M_2, 6M_\infty \# M_2,$$

and $M_\infty \# nM_2$ for each $n \geq 2$. Moreover, $2M_\infty \# nM_2$ admits Sasakian metrics of positive Ricci curvature for each $n > 0$.

Remark 1.2. From Theorem B and Table B.4.1 of [BG08], we see that the rational homology 5-spheres M_m for $m = 2, \dots, 6$ all admit a countable infinity of positive Sasaki-Seifert structures, whereas, for $m \geq 7$ only finitely many are known. Indeed, from the discussion on page 345 of [BG08] it was anticipated that M_4 admits a countable infinity of positive Sasaki-Seifert structures, but not for M_6 . Moreover, we now know that M_7 admits at least 19 positive Sasaki-Seifert structures, but whether this number is finite or not is still unknown.

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2. SASAKIAN STRUCTURES ON LINKS OF ISOLATED HYPERSURFACE SINGULARITIES

We begin with a very brief summary of the Sasakian geometry of links of isolated hypersurface singularities defined by weighted homogeneous polynomials. For more details we refer the reader to [BG08]. Consider the affine space \mathbb{C}^{n+1} together with a weighted \mathbb{C}^* -action $\mathbb{C}_{\mathbf{w}}^*$ given by $(z_0, \dots, z_n) \mapsto (\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n)$, where the *weights* w_j are positive integers. It is convenient to view the weights as the components of a vector $\mathbf{w} \in (\mathbb{Z}^+)^{n+1}$, and we shall assume that $\gcd(w_0, \dots, w_n) = 1$. Let f be a quasi-homogeneous polynomial, that is $f \in \mathbb{C}[z_0, \dots, z_n]$ and satisfies

$$f(\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \dots, z_n),$$

where $d \in \mathbb{Z}^+$ is the degree of f . We are interested in the *weighted affine cone* C_f defined by the equation $f(z_0, \dots, z_n) = 0$. We shall assume that the origin in \mathbb{C}^{n+1} is an isolated singularity, in fact the only singularity, of f . Then the *link* L_f defined by

$$L_f = C_f \cap S^{2n+1},$$

where

$$S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{j=0}^n |z_j|^2 = 1\}$$

is the unit sphere in \mathbb{C}^{n+1} , is a smooth manifold of dimension $2n - 1$. Furthermore, by the Milnor Fibration Theorem it is well-known that the link L_f is $(n - 2)$ -connected.

An important special case of weighted homogeneous polynomials, which was used to great advantage in [BGK05], is that of the *Brieskorn-Pham (BP) polynomials*. These take the form

$$f(z_0, \dots, z_n) = z_0^{a_0} + \dots + z_n^{a_n},$$

where the exponents a_j and weights w_j are related by $a_j w_j = d$ for all $j = 0, \dots, n$. The BP polynomials are the simplest type whose zero set has only an isolated singularity at the origin. There is a classification of weighted homogeneous polynomials in terms of Newton polytopes [Kou76], and a list of normal forms up to deformation has been given for $n = 2$ (3-dimensional links) by Orlik and Wagreich [OW71, OW77], and for $n = 3$ (5-dimensional links) by Yau and Yu [YY05]. It is always assumed that the polynomial is *quasi-smooth*, that is, the hypersurface in \mathbb{C}^{n+1} cut out by $f = 0$ has only an isolated singularity at the origin. The criterion for quasi-smoothness is given in [IF00] (see also Theorem 4.6.16 of [BG08]). Since this paper only deals with five dimensional links, we give this case only, viz.:

Criterion 2.1. *A hypersurface $\mathcal{Z}_d \subset \mathbb{C}\mathbb{P}(w_0, w_1, w_2, w_3)$ is quasi-smooth if and only if all of the following three conditions hold*

- (1) *For each $i = 0, \dots, 3$ there is a j and a monomial $z_i^{m_i} z_j \in H^0(\mathbb{C}\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$. Here $j = i$ is possible.*
- (2) *If $\gcd(w_i, w_j) > 1$ then there is a monomial $z_i^{b_i} z_j^{b_j} \in H^0(\mathbb{C}\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$.*

- (3) For every i, j either there is a monomial $z_i^{b_i} z_j^{b_j} \in H^0(\mathbb{C}\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$, or there are monomials $z_i^{c_i} z_j^{c_j} z_k$ and $z_i^{d_i} z_j^{d_j} z_l \in H^0(\mathbb{C}\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ with $\{k, l\} \neq \{i, j\}$.

The list of normal forms given in [YY05] consists of 19 types, referred to herein as *YY types*. It was reproduced in an appendix of [BG08], and we also reproduce it here in the appendix for the convenience of the reader. However, only 10 of the 19 types of Yau-Yu links can be written as a linear system of 4 equations in 4 unknowns. These are types I-VII, X, XI, and XIX, and are the ones we treat in this paper.

We say that a polynomial f' is a *perturbation* of a polynomial f if a monomial of the form $z_i^{a_i} z_j$ in f' with $j \neq i$ can be replaced by a monomial of the form $z_i^{a_i}$ to give f . In such a case the polynomials f and f' have the same degree d and weight vector \mathbf{w} . It means that the weight w_i of f' divides the degree d . After making all such replacements we say that f *minimizes* f' (or that f has *minimal* YY type). Once a minimal polynomial is given any additional monomials of the form $z_0^{b_0} z_1^{b_1} z_2^{b_2} z_3^{b_3}$ with $\sum_i b_i w_i = d$ can be added to f to give the most general polynomial with a given weight vector \mathbf{w} and degree d . One can then give a local moduli space of such links. In the case that Sasaki-Einstein metrics exist one then gets a local moduli space of Sasaki-Einstein metrics. See Section 5.5.2 of [BG08] for details.

Recall that a Sasakian structure consists of a quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ where g is a Riemannian metric, ξ is a unit length Killing vector field, η is a contact 1-form such that ξ is its Reeb vector field, and Φ is a $(1, 1)$ tensor field which annihilates ξ and describes an integrable complex structure on the contact vector bundle $\mathcal{D} = \ker \eta$ with transverse metric $d\eta \circ (\Phi \otimes \mathbb{1})$. On S^{2n+1} there is a well-known “weighted” Sasakian structure $\mathcal{S}_{\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$, where the vector field $\xi_{\mathbf{w}}$ is the infinitesimal generator of the circle subgroup $S_{\mathbf{w}}^1 \subset \mathbb{C}_{\mathbf{w}}^*$. This Sasakian structure on S^{2n+1} induces a natural Sasakian structure, also denoted by $\mathcal{S}_{\mathbf{w}}$, on the link L_f . We shall often write $L_f = L(\mathbf{w}; d)$ to emphasize the dependence on the weight vector and degree. The quotient space \mathcal{Z}_f of S^{2n+1} by the circle $S_{\mathbf{w}}^1$, or equivalently the space of leaves of the characteristic foliation \mathcal{F}_{ξ} of $\mathcal{S}_{\mathbf{w}}$, is a projective algebraic variety with an orbifold structure embedded in the weighted projective $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, w_1, \dots, w_n)$, in such a way that there is a commutative diagram

$$(1) \quad \begin{array}{ccc} L_f & \longrightarrow & S_{\mathbf{w}}^{2n+1} \\ \downarrow \pi & & \downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal S^1 orbundles and orbifold Riemannian submersions.

We are interested in the set of all Sasakian structures associated to the foliation \mathcal{F}_{ξ} . When the foliation is quasi-regular, this is what we have called a *Sasaki-Seifert* structure [BG08], and is conveniently phrased in terms of the basic cohomology class $[d\eta]_B \in H^{1,1}(\mathcal{F}_{\xi})$ via

$$(2) \quad \mathcal{S}^+(\mathcal{F}_{\xi}) = \{\mathcal{S}' = (a\xi, \eta', \Phi', g') \mid [d\eta']_B = a^{-1}[d\eta]_B, a > 0\}.$$

Its subset $\mathfrak{S}(\xi)$ defined by putting $a = 1$ in (2) can also be described in terms of the deformations $\eta \mapsto \eta_t = \eta + \zeta_t$ where ζ_t is basic, $\eta_0 = \eta$, $\zeta_0 = 0$ and $\eta_t \wedge (d\eta_t)^n \neq 0$ for all $t \in [0, 1]$. The *first basic Chern class* $c_1(\mathcal{F}_\xi)$ is an important invariant of the space $\mathfrak{S}(\mathcal{F}_\xi)$, and when $c_1(\mathcal{F}_\xi)$ can be represented by a positive definite $(1, 1)$ -form any Sasakian structure $\mathcal{S} \in \mathfrak{S}(\mathcal{F}_\xi)$ is said to be *positive*. For the weighted homogeneous links L_f this corresponds to $I = \sum w_i - d > 0$, and the quotient \mathcal{Z}_f being a Fano orbifold.

Suppose now we have a link L_f with a given Sasakian structure (ξ, η, Φ, g) . When can we find a 1-form ζ such that the deformed structure $(\xi, \eta + \zeta, \Phi', g')$ is Sasaki-Einstein? This is a Sasakian version of the Calabi problem and its solution is equivalent to solving the corresponding Calabi problem on the space of leaves \mathcal{Z}_f . Since a Sasaki-Einstein manifold necessarily has positive Ricci tensor, its Sasakian structure is necessarily positive. This also implies that the Kähler structure on the orbifold \mathcal{Z}_f be positive, i.e. $c_1^{orb}(\mathcal{Z}_f)$ can be represented by a positive definite $(1, 1)$ form. In this case there are well-known obstructions to solving the Calabi problem. In local uniformizing coordinates $\{z_i\}_i$ for the complex orbifold \mathcal{Z}_f , one uses the continuity method to find solutions to the Monge-Ampère equations

$$(3) \quad \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right) = \det(g_{i\bar{j}}) e^{F-t\varphi}$$

for all $t \in [0, 1]$ subject to the condition

$$(4) \quad g'_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} > 0,$$

where F is the *discrepancy potential*. See, for example, Section 5.2 of [BG08] and references therein. Generally there are obstructions to solving Equation (3) for all t . What is needed is a uniform bound. The continuity method will produce a Kähler-Einstein orbifold metric on \mathcal{Z}_f , hence, a Sasaki-Einstein metric on L_f , if there is a $\gamma > \frac{n}{n+1}$ such that for every $s \geq 1$ and for every holomorphic section $\tau_s \in H^0(\mathcal{Z}_f, \mathcal{O}(-sK_{\mathcal{Z}_f}^{orb})) = \mathcal{O}(s(|\mathbf{w}| - d))$ we have $|\tau_s|^{-\frac{\gamma}{s}} \in L^2(\mathcal{Z}_f)$, that is

$$(5) \quad \int |\tau_s|^{-\frac{2\gamma}{s}} d\text{vol}_g < +\infty.$$

Here $K_{\mathcal{Z}_f}^{orb}$ denotes the *orbifold canonical divisor* which generally differs from the ordinary canonical divisor $K_{\mathcal{Z}_f}$ by a branch divisor. Recall that a *branch divisor* Δ is a \mathbb{Q} -divisor of the form $\Delta = \sum_i (1 - \frac{1}{m_i}) D_i$ where D_i are Weil divisors on the underlying algebraic variety of \mathcal{Z}_f , and m_i is the *ramification index*, that is, the gcd of the orders of the local uniformizing groups Γ of the orbifold \mathcal{Z}_f at all points of D_i .

Fundamental to this method of producing Kähler-Einstein orbifold metrics is the work of Demailly and Kollár [DK01]. They establish the existence of Kähler-Einstein metrics in our orbifold setting unless a certain analytic multiplier ideal sheaf is non-trivial. The multiplier ideal sheaves are associated to pluri-subharmonic functions, but there is a dictionary (see pp. 176-178 of [Laz04], and pp. 157-158 of [BG08] for the orbifold setting) translating this into algebraic language. In particular, if the pluri-subharmonic function φ locally defines a divisor D on \mathcal{Z}_f that is invariant under the local uniformizing groups Γ , then triviality of the analytic multiplier ideal sheaf is equivalent to a certain algebraic condition known

as *Kawamata log-terminal (klt)*. More precisely, let $D = \sum_i a_i D_i$ be described by local Γ -invariant holomorphic functions g_i . Then the function $\varphi_D = \sum_i a_i \log |g_i|$ is plurisubharmonic and defines a multiplier ideal sheaf

$$\mathcal{J}(\varphi_D)_{\tilde{U}} = \left\{ f \in \mathcal{O}_X(\tilde{U})^\Gamma \mid \int_{\tilde{U}} \frac{|f|^2}{\prod |g_i|^{2a_i}} d\text{vol}_g < \infty \right\}$$

on the local uniformizing neighborhood \tilde{U} , and, hence, a multiplier ideal orbisheaf $\mathcal{J}(\mathcal{Z}, D)$ on \mathcal{Z}_f .

Definition 2.2. *The pair (\mathcal{Z}, D) is **klt** or **Kawamata log-terminal** if for each local uniformizing neighborhood \tilde{U} there exists a log resolution of singularities $\mu : X \rightarrow \tilde{U}$ such that*

$$K_X \equiv_n \mu^*(K_{\tilde{U}}^{\text{orb}} + \phi^*D) + \sum a_i E_i$$

with $a_i > -1$ for all μ -exceptional E_i .

This is equivalent to the non-triviality of the multiplier ideal orbisheaf, that is, the condition $\mathcal{J}(\mathcal{Z}, D) = \mathcal{O}_{\mathcal{Z}}$. Thus Demailly and Kollár prove the existence of a Kähler-Einstein orbifold metric provided all divisors numerically equivalent to an appropriate multiple of $-K_{\mathcal{Z}}^{\text{orb}}$ are klt. When phrased in terms of holomorphic sections this becomes Equation (5).

In Corollary 6.1 of [BGN03b] the authors gave the following result when the weight vector \mathbf{w} is *well-formed*, that is the gcd of any n of the weights w_0, \dots, w_n is one. This corollary is derived from Corollary 4.1 of the same paper. As has been kindly pointed out to the authors by Ivan Cheltsov, there is a gap in the proof of Corollary 4.1 as the argument sketched there only applies at the singular points of $\mathbb{P}(\mathbf{w})$ and a different technique, such as the semi-continuity argument of Johnson and Kollár given in Proposition 11 of [JK01], is required to deal with all points on \mathcal{Z}_f . So here we sketch a proof which fills this gap. Moreover, in [BGN03b] the authors only treated the case when \mathbf{w} is well-formed; however, the corrected argument works equally well in the non-well-formed case as long as one replaces the canonical divisor $K_{\mathcal{Z}_f}$ by the orbifold canonical divisor $K_{\mathcal{Z}_f}^{\text{orb}} = K_{\mathcal{Z}_f} + \Delta$ where Δ is a branch divisor, see [BGK05] and Chapter 4 of [BG08].

Lemma 2.3. *Let $L(\mathbf{w}, d)$ be a link of a weighted homogeneous hypersurface with weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3)$ ordered as $w_0 \leq w_1 \leq w_2 \leq w_3$. Let $\mathcal{Z}_{\mathbf{w}}$ denote the corresponding projective algebraic orbifold. Furthermore, let $I = |\mathbf{w}| - d$ denote the Fano index. Then*

- (1) *The 5-manifold $L(\mathbf{w}, d)$ admits a Sasaki-Einstein metric if $2Id < 3w_0w_1$.*
- (2) *If the line $z_0 = z_1 = 0$ does not lie in $\mathcal{Z}_{\mathbf{w}}$ and the weaker condition $2Id < 3w_0w_2$ holds, then $L(\mathbf{w}, d)$ admits a Sasaki-Einstein metric.*
- (3) *If the point $(0, 0, 0, 1)$ does not lie in $\mathcal{Z}_{\mathbf{w}}$ and the even weaker condition $2Id < 3w_0w_3$ holds, then $L(\mathbf{w}, d)$ admits a Sasaki-Einstein metric.*

Proof. We consider Proposition 11 of [JK01] only in the situation needed for this paper, namely where $\mathcal{Z}_{\mathbf{w}} \subset \mathbb{P}(\mathbf{w})$ is a hypersurface in weighted projective three-space. We will assume that $\mathbf{w} = (w_0, w_1, w_2, w_3)$ satisfies $w_0 \leq w_1 \leq w_2 \leq w_3$ and $\gcd(w_0, w_1, w_2, w_3) = 1$. Let $\phi : C_f \setminus \{0\} \rightarrow \mathbb{P}(\mathbf{w})$ be the natural quotient map and let $Z_i = \phi^{-1}(\mathcal{Z}_f \cap \mathbb{P}(\mathbf{w}) \setminus \{z_i = 0\})$. Proposition 11 of [JK01] states that for each $p \in Z_i$

$$\text{mult}_p(Z_i) \leq w_3w_2w_1 \deg(\mathcal{Z}_{\mathbf{w}})$$

where the degree is computed relative to $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(1)$. Moreover, as long as $\mathcal{Z}_{\mathbf{w}}$ is not the hyperplane $z_0 = 0$ then there is a better bound:

$$\text{mult}_p(Z_i) \leq w_3 w_2 w_0 \deg(\mathcal{Z}_{\mathbf{w}})$$

for all i and all $p \in Z_i$. Finally (this is the extra content of Corollary 4.1 of [BGN03b]), if $(0, 0, 0, 1)$ is not contained in $\mathcal{Z}_{\mathbf{w}}$ then for all i and all $x \in Z_i$

$$\text{mult}_p(Z_i) \leq w_2 w_1 w_0 \deg(\mathcal{Z}_{\mathbf{w}}).$$

One key idea in the proof of Proposition 11 of [JK01] is to use the affine cone C_f lying over Z_f in order to bound the multiplicity of Z_f at various points. Since the multiplicity of a variety at a point is an upper semi-continuous function of the point we deduce that for any point $p \in Z_i$

$$\text{mult}_0(C_f) \geq \text{mult}_p(Z_i)$$

and thus the goal is to bound the multiplicity of C_f at the origin.

There are two steps in the process of bounding $\text{mult}_0(C_f)$. First we cut out a zero dimensional cycle Z by successively intersecting $\mathcal{Z}_{\mathbf{w}}$ with the coordinate hyperplanes $H_i = \{z_i = 0\}$. In general, different hyperplanes are required for different components of the intersection class but in our case the entire process works globally. Lifting the cycle Z to C_f , this corresponds to cutting out the closure \bar{F}_x of each fibre $F_x = \phi^{-1}(x)$ lying over a point $x \in Z$, with ramification index r_x given by the order of the kernel of the action of \mathbf{C}^* on F_x . As to the multiplicity of these fibres at 0 which we must bound, we have

$$(6) \quad r_x \cdot \text{mult}_0(\bar{F}_x) = w_{\min_j \{z_j \neq 0\}}$$

as can be seen by looking at the relevant affine chart in the definition of $\mathbb{P}(\mathbf{w})$ as described in §1 of [JK01].

Suppose, for the moment, that the point $(0, 0, 0, 1)$ does not belong to $\mathcal{Z}_{\mathbf{w}}$. We claim that $H_0 \cap H_1 \cap \mathcal{Z}_{\mathbf{w}}$ is a proper intersection. Since $H_0 \cap H_1 \subset \mathbb{P}(\mathbf{w})$ is a line, if the intersection were not proper then the entire line $H_0 \cap H_1$ would be contained in $\mathcal{Z}_{\mathbf{w}}$ and this is not the case since $(0, 0, 0, 1)$ is not in $\mathcal{Z}_{\mathbf{w}}$. Note that a weaker hypothesis is sufficient to guarantee that $H_0 \cap H_1 \cap \mathcal{Z}_{\mathbf{w}}$ is proper but we will use the fact that $(0, 0, 0, 1)$ is not in $\mathcal{Z}_{\mathbf{w}}$ at the next step. Let x be one of the points of intersection of $H_0 \cap H_1 \cap \mathcal{Z}_{\mathbf{w}}$. Then

$$(7) \quad r_x \cdot \text{mult}_0(\bar{F}_x) \leq w_2$$

since x must be contained in one of the charts $z_i \neq 0$ for $i \leq 2$ and so (7) follows from (6). Since $\mathcal{Z}_{\mathbf{w}} \cap H_0 \cap H_1$ is a cycle of degree at most $\deg(\mathcal{Z}_{\mathbf{w}})w_0w_1$ we see from (7), lifting to C_f , that

$$\text{mult}_0(C_f) \leq \deg(\mathcal{Z}_{\mathbf{w}})w_0w_1w_2$$

as desired; indeed, the right hand side is the total degree of an intersection class, part of which, the left hand side, is supported at 0. The other two inequalities of [JK01] Proposition 11 are obtained similarly except that in these cases one cannot necessarily choose H_0 and H_1 for the intersection and one does not necessarily have the inequality (6) and hence the inequalities obtained are weaker. This proves the lemma as well as fills the gap in Corollary 6.1 of [BGN03b]. \square

Note that all BP polynomials satisfy the better estimate (iii) of Lemma 2.3. However, in [BGK05] a different estimate was used for the case of BP polynomials. These two estimates are somewhat independent. The estimate used in [BGK05] appears to be the better estimate for spheres, but the estimate in (iii) of Lemma 2.3 seems to be the better for Smale manifolds of mixed type as well as some rational homology spheres.

We also remark that many of the Sasaki-Einstein metrics given here occur with ‘moduli’. Using methods explained in [BG08], it is straightforward to compute infinitesimal deformations of a Sasaki-Einstein structure on a given link. However, completeness of this infinitesimal deformation space is still unknown, and will be treated elsewhere.

3. SMALE MANIFOLDS

The reason that dimension five is so amenable to analysis stems from Smale’s seminal work [Sma62] on the classification of compact simply connected spin 5-manifolds. The non-spin case was later completed by Barden [Bar65], but the spin case will suffice here. Smale’s classification of all closed simply connected 5-manifolds that admit a spin structure is as follows. Any such manifolds must be of the form

$$(8) \quad M = kM_\infty \# M_{m_1} \# \cdots \# M_{m_n}$$

where $M_\infty = S^2 \times S^3$, kM_∞ is the k -fold connected sum of M_∞ , m_i is a positive integer with m_i dividing m_{i+1} and $m_1 \geq 1$, and where M_m is S^5 if $m = 1$, or a 5-manifold such that $H_2(M_m, \mathbb{Z}) = \mathbb{Z}/m \oplus \mathbb{Z}/m$, otherwise. Here $k \in \mathbb{N}$ and $k = 0$ means that there is no M_∞ factor at all. It will also be convenient to use the convention $0M_\infty = S^5$, which is consistent with the fact that the sphere is the identity element for the connected sum operation.

The understanding of Sasakian geometry in dimension five owes much to the work of Kollár [Kol05, Kol06, Kol08]. In particular, he has given a complete characterization of the torsion [Kol05] for smooth Seifert bundles over a projective cyclic orbifold surface. Explicitly, he has shown that if a Smale manifold M admits a Sasakian structure, then $H_2(M, \mathbb{Z})$ must take the form

$$(9) \quad H_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus \sum_i (\mathbb{Z}_{m_i})^{2g(D_i)},$$

where $g(D)$ denotes the genus of the branch divisor D . This shows that only non-rational branch divisors contribute to torsion. To determine the Smale manifold we can also use a combinatorial method due to Orlik [Orl72] and a computer program written for us by E. Thomas. We refer to Corollary 9.3.13 and Conjecture 9.3.15 of [BG08] for complete details. This conjecture is known to hold in dimension five [BGS08].

4. TORSION AND BRANCH DIVISORS

Heretofore, $L_f = L_f(\mathbf{w}, d)$ will denote a 5 dimensional link of a weighted homogeneous polynomial f of degree d and weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3)$. As mentioned previously such five dimensional links have been classified by Yau and Yu [YY05]. They gave a list of 19 normal forms which is reproduced in the Appendix. First we make note of the fact that 10 of the 19 types of Yau-Yu links can

be written as a linear system of 4 equations in 4 unknowns. These are types I-VII, X,XI, and XIX. All of these can be written as a matrix equation in the form

$$(10) \quad \mathbf{A} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} = d \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

where \mathbf{A} is 4 by 4 matrix depending on the exponents \mathbf{a} . We refer to \mathbf{A} as the *exponent matrix*. We assume that $\det \mathbf{A} \neq 0$, in which case we write the solution as $\mathbf{w} = d\mathbf{A}^{-1}\mathbf{1}_4$. We refer to links of this type as *standard YY links*. We are interested in 5-manifolds M^5 that have non-zero torsion, and by Kollár's result Equation (9) above, non-trivial torsion implies there must be branch divisors that are non-rational curves. We now consider some useful lemmas.

Lemma 4.1. *Let $\mathcal{Z}_{\mathbf{w}}$ be the algebraic orbifold of a link $L_f = L_f(\mathbf{w}, d)$. Then any branch divisor D that contributes to torsion in $H_2(L_f, \mathbb{Z})$ is a coordinate hypersurface in $\mathcal{Z}_{\mathbf{w}}$, that is, D is the hypersurface $z_i = 0$ for some $i = 0, \dots, 3$. In particular, $\mathcal{Z}_{\mathbf{w}}$ can have at most four branch divisors contributing to torsion.*

Proof. This follows from Kollár's result (9) above and the discussion on page 141 of [BG08]. Indeed, if D is a branch divisor contributing to torsion then the algebraic orbifold $\mathcal{Z}_{\mathbf{w}}$ is not well-formed, so some $d_i = \gcd(w_0, \dots, \hat{w}_i, \dots, w_3) > 1$, and D is obtained by setting $z_i = 0$. \square

Lemma 4.2. *Let $L_f = L_f(\mathbf{w}; d)$ be a link such that the polynomial f contains a term of the form $z_j^{a_j} z_i$. Then D_i is not a branch divisor.*

Proof. Since the polynomial f contains a term of the form $z_i z_j^{a_j}$, we must have $d = a_j w_j + w_i$. But if D_i were a branch divisor, we would have $m_i = \gcd(w_j : j \neq i) > 1$. But then since $m_i | d$ and all the w_j with $j \neq i$, it must divide $w_i = d - a_j w_j$ as well. But this is impossible since $\gcd(w_0, w_1, w_2, w_3) = 1$. \square

Using Lemmas 4.1 and 4.2 we give a table of the number N of possible branch divisors that can contribute to torsion for each of the standard YY types.

YY type	I	II	III	IV	V	VI	VII	X	XI	XIX
N	4	3	2	2	1	0	2	1	1	0

Moreover, it follows from Proposition 6.3 of [Kol05] that a positive Sasakian structure can have at most one non-rational branch divisor.

The following result about 3-dimensional links will prove useful.

Lemma 4.3. *Any curve C of the form $z_0^{a_0} + z_1^{a_1} + z_1 z_2 = 0$ or $z_0^{a_0} + z_1 z_2 = 0$ is rational.*

Proof. In both cases we have $d = w_1 + w_2$. So $|\mathbf{w}| - d = w_0 > 0$, which implies that C is rational by a theorem of Orlik [Orl70]. \square

Instead of presenting a systematic investigation of all the YY links which would require computer searches, we prefer here to give some methods of ferreting out links with torsion in the hope of providing at least partial solutions to the problems given in the introduction. We also restrict ourselves at this time to the standard YY links. A more systematic approach using computer searches is currently in

progress. Here we present two methods which are complementary in the sense that one begins with the case of many (meaning four) branch divisors, namely links of BP polynomials (which are classified in [BG08]) with fixed exponents and relatively high index and changes (a procedure we call *jiggling*) the weight vector to obtain a different YY type. The second method begins with a YY type with few branch divisors, and tries to give conditions to guarantee non-trivial torsion. This can also give links with more branch divisors as ‘degenerate cases’. The minimal YY type is always presented. We illustrate these procedures with examples.

Example 4.4. Consider the BP link of degree 24 and exponent vector $\mathbf{a} = (2, 3, 8, 8)$ given in Table B.4.1 of [BG08]. This gives a positive Sasaki-Seifert structure on the rational homology sphere $3M_3$ which also admits a Sasaki-Einstein metric. Its weight vector is $\mathbf{w} = (12, 8, 3, 3)$, and its degree is $d = 24$. We see that D_2 is a branch divisor with ramification index 3 and genus 3. Since the index $I = |\mathbf{w}| - d = 2$, we can jiggle the weight vector by lowering the weight w_1 to 7 and replacing the monomial z_1^3 by $z_1^3 g_1(z_2, z_3)$ where g_1 is a linear polynomial in z_2 and z_3 . This gives the new link of YY type II with weight vector $\mathbf{w} = (12, 7, 3, 3)$, degree $d = 24$, and index $I = 1$ on the Smale manifold $M_\infty \# 3M_3$. Moreover, one sees by Lemma 2.3 that this admits a Sasaki-Einstein metric.

Example 4.5. For the second procedure we consider a YY link of type X. In this case the normal form is

$$z_0^{a_0} + z_1^{a_1} z_2 + z_2^{a_2} z_3 + z_1 z_3^{a_3}.$$

The positivity condition is

$$\frac{1}{a_0} + \frac{a_1 a_2 + a_1 a_3 + a_2 a_3 - a_1 - a_2 - a_3 + 3}{a_1 a_2 a_3 + 1} > 1.$$

The only possible branch divisor is D_0 which we want to be a non-rational curve. Notice that if we denote the degree of the divisor D_0 by d' with weights $\mathbf{w}' = (w'_1, w'_2, w'_3)$, then a necessary condition for D_0 to be non-rational is $d' - |\mathbf{w}'| \geq 0$. Now we write $\mathbf{w} = (w_0, mw'_1, mw'_2, mw'_3)$ with $\gcd(w'_1, w'_2, w'_3) = 1$. We also have

$$(11) \quad d = a_0 w_0 = a_1 m w'_1 + m w'_2 = a_2 m w'_2 + m w'_3 = a_3 m w'_3 + m w'_1.$$

So m divides d and $\gcd(m, w_0) = 1$, so we write $d = m d'$ and $a_0 = m a'_0$. The general solution for the weight vector \mathbf{w}' of the branch divisor D_0 is

$$(12) \quad \mathbf{w}' = \begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \end{pmatrix} = \frac{d'}{1 + a_1 a_2 a_3} \begin{pmatrix} 1 - a_3 + a_2 a_3 \\ 1 - a_1 + a_1 a_3 \\ 1 - a_2 + a_1 a_2 \end{pmatrix}.$$

We see from Equation (11) that $\gcd(w'_i, d') = 1$ and $\gcd(w'_i, w'_j) = 1$ for all i, j . Moreover, from (12) and the fact that w'_i are relatively prime integers, we must have $q d' = 1 + a_1 a_2 a_3$ where q is a positive integer that must divide $b_i := a_i a_{i+1} - a_{i+1} + 1$ where $i \in \{1, 2, 3\} \pmod{3}$. Now using the genus formula (Equation 4.6.5 of [BG08]), and the general solution (12) gives, with the help of Maple,

$$(13) \quad g(D_0) = \frac{1 + a_1 a_2 a_3 - d'}{2d'} = \frac{q - 1}{2}$$

which must be a non-negative integer implying that q must be odd.

We consider the case $\mathbf{a} = (2, 3, 5, a_3)$ which also implies $m = 2$. We look for solutions with $q > 1$ and odd, of course. It is now quite easy to satisfy the positivity condition which gives $5 \leq a_3 \leq 18$. Now searching for possible b_i , we see that the

only solution is for $a_3 = 8$, in which case we have $b_1 = 11, b_2 = 33$ and $b_3 = 22$ giving $q = 11$. We find $d = 22$ and $\mathbf{w} = (2, 4, 6, 11)$ which is a perturbation of type VII, and occurs on the manifold $5M_2$. It satisfies the optimal klt condition of Lemma 2.3, and thus, admits a Sasaki-Einstein metric.

5. PROOFS OF THEOREMS A AND B

Given our previous discussion the proofs of the main theorems are now straightforward. We need only exhibit links $L(\mathbf{w}, d)$ of isolated hypersurface singularities by weighted homogenous polynomials of degree d and weight vector \mathbf{w} , and then check the positivity condition $I = |\mathbf{w}| - d > 0$, and the klt estimates of Lemma 2.3. We list the results in the appendix in the form of tables. Since we have come up short of a complete classification, we do not bother keeping track of the number of Sasaki-Seifert structures that occur when the cardinality is finite. We do, however, record the cases when there are a countable infinity of Sasaki-Seifert structures.

6. APPENDIX: TABLES

Here we present tables giving links of isolated hypersurface singularities by weighted homogeneous polynomials. In the tables we list the weight vector \mathbf{w} , the degree d , the Fano index $I = |\mathbf{w}| - d$ of the weighted homogeneous polynomial, the manifold, and whether there is a Sasaki-Einstein metric. In the case of the sporadic solution, we also list the exponent vector \mathbf{a} of a typical polynomial as well as the Yau-Yu normal form type [YY05] (see also Table B.5 of [BG08]). We remark that the exponent vector is not generally an invariant of the Sasaki-Seifert structure, nor of the link, so we list the \mathbf{a} that ‘minimizes’ the YY type. Moreover, we order the weights as $w_0 \leq w_1 \leq w_2 \leq w_3$.

First we exhibit a table of infinite series solutions that admit Sasakian metrics of positive Ricci curvature, and will thus prove Theorem B. Four of the series admit Sasaki-Einstein metrics proving part (3) of Theorem A, while 3 series cannot admit Sasaki-Einstein metrics when $k > 4$. The existence of such metrics is unknown in all other cases. It is interesting that the only series solutions that we found that admit Sasaki-Einstein metrics all involve M_3 . Note that here $(a, b) = \gcd(a, b)$.

weight vector \mathbf{w}	d	I	manifold	SE
$(2, 2k, k(2n+1), 2k(n+1)), (k, 2) = 1$	$4k(n+1)$	$k+2$	$M_\infty \# nM_2$	<i>no, if $k > 4$</i>
$(2, 4, 4n, 4n+1)$	$4(2n+1)$	3	$2M_\infty \# nM_2$?
$(4, 4k+2, 4k+3, 2(4k+3))$	$4(4k+3)$	3	$2M_\infty \# M_2$?
$(4, 4k-1, 4k, 8k)$	$16k$	3	$M_\infty \# M_4$?
$(2, 3k+2, 4k+2, 2(3k+2)), (k, 2) = 1$	$4(3k+2)$	$k+2$	$3M_\infty \# M_2$	<i>no, if $k > 4$</i>
$(3, 3k+2, 3(2k+1), 3(3k+2))$	$6(3k+2)$	2	$3M_\infty \# M_3$	<i>yes</i>
$(6, 3(2k+1), 4(3k+1), 9(2k+1))$	$18(2k+1)$	4	$M_\infty \# M_3$	<i>yes</i>
$(3, 3k, 6k-1, 9k)$	$18k$	2	$M_\infty \# 2M_3$	<i>yes</i>
$(3, 3k-1, 3k, 3k)$	$9k$	2	$2M_\infty \# M_3$?
$(2, 2k+1, 2(2k+1), 2(3k+1))$	$6(2k+1)$	1	$6M_\infty \# M_2$?
$(2, 2k, 4k, 6k-1)$	$12k$	1	$2M_\infty \# 4M_2$?
$(6, 2k, 4k, 3(2k-1)), (k, 3) = 1$	$12k$	3	$2M_\infty \# M_2$?
$(2, 2k, 2k, 4k-1)$	$8k$	1	$3M_\infty \# 3M_2$?
$(3, 3k, 6k-1, 9k)$	$18k$	2	$2M_3$	<i>yes</i>
$(4, 3(2k+1), 4(2k+1), 4(3k+1))$	$12(2k+1)$	$2k+3$	M_4	<i>no, if $k > 4$</i>
$(6, 6k-1, 12k, 18k)$	$36k$	5	M_6	?

Table 1. Series solutions giving a countable infinity of positive Sasaki-Seifert structures on manifolds having non-vanishing torsion.

The next two tables will complete the proof of Theorem A. The first table gives a list of rational homology spheres of the form nM_2 that can admit Sasaki-Einstein metrics. None of these rational homology spheres with $n > 1$ were known previously to admit such metrics. We also indicate some links where the existence of Sasaki-Einstein metrics is still unknown.

a	w	<i>d</i>	<i>I</i>	manifold	<i>SE</i>	YYtype
(11, 5, 3, 2)	(2, 4, 6, 11)	22	1	$5M_2$	<i>yes</i>	<i>VII</i>
(7, 7, 3, 2)	(2, 2, 4, 7)	14	1	$6M_2$	<i>yes</i>	<i>II</i>
(27, 7, 3, 2)	(4, 18, 42, 63)	126	1	$3M_2$	<i>yes</i>	<i>II</i>
(14, 9, 3, 2)	(8, 14, 42, 63)	126	1	M_2	<i>yes</i>	<i>II</i>
(9, 7, 3, 2)	(4, 6, 14, 21)	42	3	$3M_2$?	<i>II</i>
(9, 7, 3, 2)	(4, 6, 14, 19)	42	1	$3M_2$	<i>yes</i>	<i>VII</i>
(11, 9, 3, 2)	(18, 20, 66, 99)	198	5	M_2	?	<i>II</i>
(11, 9, 3, 2)	(18, 22, 60, 99)	198	1	M_2	<i>yes</i>	<i>II</i>
(10, 9, 3, 2)	(8, 10, 30, 45)	90	3	M_2	<i>yes</i>	<i>II</i>
(10, 9, 3, 2)	(6, 10, 30, 45)	90	1	M_2	<i>yes</i>	<i>II</i>
(15, 7, 3, 2)	(2, 4, 10, 15)	30	1	$6M_2$	<i>yes</i>	<i>II</i>
(12, 9, 3, 2)	(4, 6, 18, 27)	54	1	$3M_2$	<i>yes</i>	<i>II</i>
(17, 4, 3, 2)	(4, 14, 26, 41)	82	3	$2M_2$?	<i>X</i>
(12, 4, 3, 2)	(4, 10, 18, 29)	58	3	$2M_2$?	<i>X</i>
(7, 4, 3, 2)	(4, 6, 10, 17)	34	3	$2M_2$?	<i>X</i>
(15, 5, 3, 2)	(2, 6, 8, 15)	30	1	$5M_2$	<i>yes</i>	<i>II</i>
(5, 5, 5, 2)	(2, 2, 2, 5)	10	1	$6M_2$	<i>yes</i>	<i>I</i>
(6, 5, 5, 2)	(4, 6, 6, 15)	30	1	$2M_2$	<i>yes</i>	<i>II</i>
(7, 5, 5, 2)	(10, 12, 14, 35)	70	1	$2M_2$	<i>yes</i>	<i>II</i>
(9, 9, 3, 2)	(2, 2, 6, 9)	18	1	$7M_2$	<i>yes</i>	<i>I</i>
(21, 7, 3, 2)	(2, 6, 14, 21)	42	1	$6M_2$	<i>yes</i>	<i>I</i>
(15, 9, 3, 2)	(6, 10, 30, 45)	90	1	M_2	<i>yes</i>	<i>I</i>

Table 2. Sporadic examples of rational homology spheres of the form nM_2 with $n > 1$ that can admit a Sasaki-Einstein metric.

Finally, we give a table of Smale manifolds of mixed type most of which admit Sasaki-Einstein metrics, thus completing the proof of Theorem A.

a	w	<i>d</i>	<i>I</i>	manifold	<i>SE</i>	<i>YYtype</i>
(5, 3, 3, 3)	(9, 10, 12, 15)	45	1	$M_\infty \# M_3$	<i>yes</i>	<i>IV</i>
(8, 4, 3, 2)	(3, 6, 7, 9)	24	1	$M_\infty \# 2M_3$	<i>yes</i>	<i>IV</i>
(9, 4, 3, 2)	(2, 4, 6, 7)	18	1	$2M_\infty \# 3M_2$?	<i>VII</i>
(10, 4, 3, 2)	(4, 10, 12, 15)	40	1	$3M_\infty \# M_2$	<i>yes</i>	<i>IV</i>
(6, 4, 4, 2)	(8, 9, 12, 20)	48	1	$M_\infty \# M_4$	<i>yes</i>	<i>IV</i>
(16, 5, 3, 2)	(3, 9, 13, 24)	48	1	$M_\infty \# 2M_3$	<i>yes</i>	<i>VII</i>
(6, 4, 2, 2)	(4, 6, 9, 10)	24	5	$M_\infty \# M_2$?	<i>IV</i>
(6, 4, 4, 2)	(4, 6, 6, 9)	24	1	$M_\infty \# M_2$	<i>yes</i>	<i>IV</i>
(6, 5, 3, 2)	(10, 12, 16, 25)	60	3	$M_\infty \# M_2$?	<i>IV</i>
(12, 5, 3, 2)	(10, 24, 32, 55)	120	1	$M_\infty \# M_2$	<i>yes</i>	<i>IV</i>
(6, 4, 3, 3)	(4, 5, 8, 8)	24	1	$2M_\infty \# M_4$	<i>yes</i>	<i>II</i>
(8, 6, 4, 2)	(6, 7, 12, 24)	48	1	$M_\infty \# M_6$	<i>yes</i>	<i>II</i>
(12, 4, 4, 2)	(4, 9, 12, 24)	48	1	$M_\infty \# M_4$	<i>yes</i>	<i>II</i>
(7, 6, 4, 2)	(6, 7, 9, 21)	42	1	$2M_\infty \# M_3$	<i>yes</i>	<i>II</i>
(6, 6, 4, 2)	(4, 4, 5, 12)	24	1	$M_\infty \# 2M_4$	<i>yes</i>	<i>II</i>
(12, 5, 4, 2)	(5, 11, 15, 30)	60	1	$M_\infty \# M_5$	<i>yes</i>	<i>II</i>
(12, 7, 3, 2)	(7, 8, 28, 42)	84	1	$M_\infty \# M_7$	<i>yes</i>	<i>II</i>
(10, 8, 3, 2)	(4, 5, 12, 20)	40	1	$3M_\infty \# M_4$	<i>yes</i>	<i>II</i>
(8, 8, 3, 2)	(3, 3, 7, 12)	24	1	$M_\infty \# 3M_3$	<i>yes</i>	<i>II</i>
(21, 6, 3, 2)	(2, 7, 14, 20)	42	1	$6M_\infty \# M_2$?	<i>II</i>
(8, 4, 4, 2)	(2, 3, 4, 8)	16	1	$5M_\infty \# M_2$	<i>yes</i>	<i>II</i>
(6, 6, 4, 2)	(2, 2, 3, 6)	12	1	$5M_\infty \# 2M_2$	<i>yes</i>	<i>I</i>
(12, 8, 3, 2)	(2, 3, 8, 12)	24	1	$6M_\infty \# M_2$	<i>yes</i>	<i>I</i>
(10, 5, 4, 2)	(2, 4, 5, 10)	20	1	$4M_\infty \# 2M_2$	<i>yes</i>	<i>I</i>
(12, 9, 3, 2)	(3, 4, 12, 18)	36	1	$4M_\infty \# M_3$	<i>yes</i>	<i>I</i>
(8, 6, 4, 2)	(3, 4, 6, 12)	24	1	$3M_\infty \# M_3$	<i>yes</i>	<i>I</i>
(12, 10, 3, 2)	(5, 6, 20, 30)	60	1	$2M_\infty \# M_5$	<i>yes</i>	<i>I</i>
(18, 8, 3, 2)	(4, 9, 24, 36)	72	1	$2M_\infty \# M_4$	<i>yes</i>	<i>I</i>
(9, 6, 4, 2)	(4, 6, 9, 18)	36	1	$2M_\infty \# M_2$	<i>yes</i>	<i>I</i>
(12, 6, 3, 2)	(3, 6, 10, 18)	36	1	$M_\infty \# 2M_3$	<i>yes</i>	<i>II</i>
(12, 6, 3, 2)	(3, 6, 11, 18)	36	2	$M_\infty \# 2M_3$	<i>yes</i>	<i>II</i>
(6, 6, 3, 2)	(3, 3, 5, 9)	18	2	$M_\infty \# 2M_3$	<i>yes</i>	<i>II</i>
(20, 5, 4, 2)	(3, 12, 16, 30)	60	1	$M_\infty \# 2M_3$	<i>yes</i>	<i>II</i>
(8, 5, 4, 2)	(5, 7, 10, 20)	40	2	$M_\infty \# M_5$	<i>yes</i>	<i>II</i>
(8, 5, 4, 2)	(5, 6, 10, 20)	40	1	$M_\infty \# M_5$	<i>yes</i>	<i>II</i>

Table 3. Some new sporadic examples of simply connected 5-dimensional manifolds of mixed type which can admit Sasaki-Einstein metrics.

Table: The Yau-Yu Links in Dimensions 5		
Type	$f(z_0, z_1, z_2, z_3)$	$ w /\text{degree}(f)$
I (BP)	$z_0^a + z_1^b + z_2^c + z_3^d$	$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$
II	$z_0^a + z_1^b + z_2^c + z_2 z_3^d$	$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{c-1}{cd}$
III	$z_0^a + z_1^b + z_2^c z_3 + z_2 z_3^d$	$\frac{1}{a} + \frac{1}{b} + \frac{d-1}{cd-1} + \frac{c-1}{cd-1}$
IV	$z_0^a + z_0 z_1^b + z_2^c + z_2 z_3^d$	$\frac{1}{a} + \frac{a-1}{ab} + \frac{1}{c} + \frac{c-1}{cd}$
V	$z_0^a z_1 + z_0 z_1^b + z_2^c + z_2 z_3^d$	$\frac{b-1}{ab-1} + \frac{a-1}{ab-1} + \frac{1}{c} + \frac{c-1}{cd}$
VI	$z_0^a z_1 + z_0 z_1^b + z_2^c z_3 + z_2 z_3^d$	$\frac{b-1}{ab-1} + \frac{a-1}{ab-1} + \frac{d-1}{cd-1} + \frac{c-1}{cd}$
VII	$z_0^a + z_1^b + z_1 z_2^c + z_2 z_3^d$	$\frac{1}{a} + \frac{1}{b} + \frac{b-1}{bc} + \frac{[b(c-1)+1]}{bcd}$
VIII	$z_0^a + z_1^b + z_1 z_2^c + z_1 z_3^d + z_2^p z_3^q$ $\frac{p(b-1)}{bc} + \frac{q(b-1)}{bd} = 1$	$\frac{1}{a} + \frac{1}{b} + \frac{b-1}{bc} + \frac{b-1}{bd}$
IX	$z_0^a + z_1^b z_3 + z_2^c z_3 + z_1 z_3^d + z_1^p z_2^q$ $\frac{p(d-1)}{bd-1} + \frac{q(b-1)}{c(bd-1)} = 1$	$\frac{1}{a} + \frac{(d-1)}{bd-1} + \frac{b(d-1)}{c(bd-1)} + \frac{b-1}{bd-1}$
X	$z_0^a + z_1^b z_2 + z_2^c z_3 + z_1 z_3^d$	$\frac{1}{a} + \frac{[d(c-1)+1]}{bcd+1} + \frac{[b(d-1)+1]}{bcd+1} + \frac{[c(b-1)+1]}{bcd+1}$
XI	$z_0^a + z_0 z_1^b + z_1 z_2^c + z_2 z_3^d$	$\frac{1}{a} + \frac{a-1}{ab} + \frac{[a(b-1)+1]}{[ab(c-1)+(a-1)]} + \frac{abcd}{abcd}$
XII	$z_0^a + z_0 z_1^b + z_0 z_2^c + z_1 z_3^d + z_1^p z_2^q$ $\frac{p(a-1)}{ab} + \frac{q(a-1)}{ac} = 1$	$\frac{1}{a} + \frac{a-1}{ab} + \frac{a-1}{ac} + \frac{[a(b-1)+1]}{abd}$
XIII	$z_0^a + z_0 z_1^b + z_1 z_2^c + z_1 z_3^d + z_2^p z_3^q$ $\frac{p[a(b-1)+1]}{abc} + \frac{q[a(b-1)+1]}{abd} = 1$	$\frac{1}{a} + \frac{a-1}{ab} + \frac{a-1}{ac} + \frac{[a(b-1)+1]}{abd}$
XIV	$z_0^a + z_0 z_1^b + z_0 z_2^c + z_0 z_3^d + z_1^p z_2^q + z_2^r z_3^s$ $\frac{p(a-1)}{ab} + \frac{q(a-1)}{ac} = 1 = \frac{r(a-1)}{ac} + \frac{s(a-1)}{ad}$	$\frac{1}{a} + \frac{a-1}{ab} + \frac{a-1}{ac} + \frac{a-1}{ad}$
XV	$z_0^a z_1 + z_0 z_1^b + z_0 z_2^c + z_2 z_3^d + z_1^p z_2^q$ $\frac{p(a-1)}{ab-1} + \frac{q(b-1)}{c(ab-1)} = 1$	$\frac{b-1}{ab-1} + \frac{a-1}{ab-1} + \frac{b(a-1)}{c(ab-1)} + \frac{[c(ab-1)-b(a-1)]}{cd(ab-1)}$
XVI	$z_0^a z_1 + z_0 z_1^b + z_0 z_2^c + z_0 z_3^d + z_1^p z_2^q + z_2^r z_3^s$ $\frac{p(a-1)}{ab-1} + \frac{q(b-1)}{c(ab-1)} = 1 = \frac{r(a-1)}{ac} + \frac{s(a-1)}{ad}$	$\frac{(b-1)}{ab-1} + \frac{(a-1)}{ab-1} + \frac{b(a-1)}{c(ab-1)} + \frac{b(a-1)}{d(ab-1)}$
XVII	$z_0^a z_1 + z_0 z_1^b + z_1 z_2^c + z_0 z_3^d + z_1^p z_3^q + z_0^r z_2^s$ $\frac{p(a-1)}{ab-1} + \frac{q(b-1)}{d(ab-1)} = 1 = \frac{r(b-1)}{ab-1} + \frac{sa(b-1)}{c(ab-1)}$	$\frac{b-1}{ab-1} + \frac{a-1}{ab-1} + \frac{a(b-1)}{c(ab-1)} + \frac{b(a-1)}{d(ab-1)}$
XVIII	$z_0^a z_2 + z_0 z_1^b + z_1 z_2^c + z_1 z_3^d + z_2^p z_3^q$ $\frac{p[a(b-1)+1]}{abc+1} + \frac{q[a(b-1)+1]}{d(abc+1)} = 1$	$\frac{[b(c-1)+1]}{abc+1} + \frac{[c(a-1)+1]}{abc+1} + \frac{[a(b-1)+1]}{c(abc+1)} + \frac{[a(b-1)+1]}{d(abc+1)}$
XIX	$z_0^a z_3 + z_0 z_1^b + z_2^c z_3 + z_2 z_3^d$	$\frac{[b(d(c-1)+1)-1]}{abcd-1} + \frac{[d(c(a-1)+1)-1]}{abcd-1} + \frac{[a(b(d-1)+1)-1]}{abcd-1} + \frac{[c(a(b-1)+1)-1]}{abcd-1}$

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