TRANSVERSE KÄHLER HOLONOMY IN SASAKI GEOMETRY AND $\mathcal{S}$-STABILITY

CHARLES P. BOYER, HONGNIAN HUANG,
AND CHRISTINA W. TØNNESEN-FRIEDMAN

Abstract. We study the transverse Kähler holonomy groups on Sasaki manifolds $(M, \mathcal{S})$ and their stability properties under transverse holomorphic deformations of the characteristic foliation by the Reeb vector field. In particular, we prove that when the first Betti number $b_1(M)$ and the basic Hodge number $h^{0,2}_B(\mathcal{S})$ vanish, then $\mathcal{S}$ is stable under deformations of the transverse Kähler flow. In addition we show that an irreducible transverse hyperkähler Sasakian structure is $\mathcal{S}$-unstable, whereas, an irreducible transverse Calabi-Yau Sasakian structure is $\mathcal{S}$-stable when $\dim M \geq 7$. Finally, we prove that the standard Sasaki join operation (transverse holonomy $U(n_1) \times U(n_2)$) as well as the fiber join operation preserve $\mathcal{S}$-stability.

1. Introduction

It is well known from Berger's classification of Riemannian holonomy that the irreducible holonomy groups in Kähler geometry are precisely, $U(n), SU(n)$ and $Sp(n)$ which correspond to irreducible Kähler, Calabi-Yau, and hyperkähler geometry, respectively. There is also a well known Stability Theorem of Kodaira and Spencer [KS60] that says that any infinitesimal deformation of a compact complex manifold which is Kähler remains Kähler. A similar result was obtained in the other two cases by Goto [Got04]. Analogues of these stability theorems for holomorphic foliations was proven by El Kacimi Alaoui and Gmira in [EKAG97] in the Kähler case, and by Moriyama [Mor10] in the Calabi-Yau case. See also [TV08]. The two special holonomy cases have been studied further by Habib and Vezzoni [HV15]. In particular, they prove that a transverse Kähler foliation admits a transverse hyperkähler structure if and only if it admits a transverse hyperhermitian

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structure. This is the transverse version of a result of Verbitsky [Ver05] in the compact Kähler manifold case.

The purpose of this paper is to study these transverse versions and their relationship to Sasaki geometry. It is well known [KS60] that there are obstructions, namely the Hodge numbers $h^{0,2}$, for deformations of projective algebraic structures to remain projective algebraic. The point is that the transverse Kähler structure of a Sasakian structure is algebraic in an appropriate sense, cf. Section 7.5 of [BG08]. A Sasakian structure $\mathcal{S}$ is said to be $\mathcal{S}$-stable (or $\mathcal{S}$-rigid) if every sufficiently small transverse Kählerian deformation of $\mathcal{S}$ remains Sasakian. So an important question is

Question 1.1. *Which Sasakian structures are $\mathcal{S}$-stable and which are $\mathcal{S}$-unstable?*

It was recently shown by Nozawa [Noz14] that Sasaki nilmanifolds of dimension at least 5 are $\mathcal{S}$-unstable, that is, their transverse Kähler deformations become non-algebraic. This is done by deforming the transverse Kähler flow on a Sasaki nilmanifold, i.e on the total space of an S^1 bundle over an Abelian variety of complex dimension at least two. These nilmanifolds are discussed briefly in Section 4.3. Recent work of Goertsches, Nozawa, and Töben [GNT16] shows that positive Sasakian structures and toric Sasakian structures are $\mathcal{S}$-stable. Building on results of Nozawa [Noz14] we obtain the first main result of this paper:

Theorem 1.2. *Let $(M,\mathcal{S})$ be a Sasaki manifold with vanishing first Betti number and such that the basic Hodge numbers satisfy $h^{0,2}_B = h^{2,0}_B = 0$. Then $\mathcal{S}$ is $\mathcal{S}$-stable.*

Moriyama's Stability Theorem for irreducible transverse Calabi-Yau structures follows as a special case of Theorem 1.2.

Corollary 1.3. *Let $(M,\mathcal{S})$ be a Sasaki manifold of dimension $2n+1$ with $n>2$ and transverse holonomy group equal to $SU(n)$ Then $(M,\mathcal{S})$ is $\mathcal{S}$-stable. Moreover, the local universal deformation space is isomorphic to an open set in $H^1(M,\Theta)$.*

For irreducible transverse hyperkähler geometry the contrary holds which gives the second main result of the paper.

Theorem 1.4. *Let $(M,\mathcal{S})$ be a Sasaki manifold with a compatible irreducible transverse hyperkähler structure. Then $(M,\mathcal{S})$ is $\mathcal{S}$-unstable. Moreover, the local universal deformation space is isomorphic to an open set in $H^1(M,\Theta)$.*

Much more can be said about irreducible transverse hyperkähler structures in dimension 5. First, a classification of simply connected 5-manifolds that admit null Sasakian structures has just recently been completed [CMST20] by proving the existence of orbifold K3 surfaces X with second Betti number $b_2(X) = 3$. This completes the classification initiated in [BGM06, BG08, CV14]. A simply connected 5-manifold which admits a null Sasakian structure is diffeomorphic to a k-fold connected sum

$$\#k(S^2 \times S^3) \quad \text{with } k = 2, \dots, 21 \tag{1}$$

and each such 5-manifold admits a null Sasakian structure. These are represented as S^1 orbibundles over $K3$ orbifolds X_k with $b_2(X_k) = k + 1$ and $\pi_1^{orb}(X_k) = \mathbb{1}$. A smooth K3 surface is diffeomorphic to X_{22}. Furthermore, any 5-manifold of the form $\#k(S^2 \times S^3)$ admits positive Sasakian structures (cf. Corollary 11.4.8 of [BG08]) which are stable by [Noz14], so Theorems 1.2 and 1.4 give

Corollary 1.5. *Any null Sasakian structure on a simply connected 5-manifold M is $\mathcal{S}$-unstable, and all such M are of the form of Equation* (1). *So the manifolds $\#k(S^2 \times S^3)$ with $k = 2, \dots, 21$ admit both $\mathcal{S}$-stable and $\mathcal{S}$-unstable Sasakian structures.*

Given this corollary and Nozawa's result for Sasaki nilmanifolds one might wonder whether every null Sasakian structure is $\mathcal{S}$-unstable. But this is not true for non-trivial S^1 bundles over an Enriques surfaces E even though these are smooth $\mathbb{Z}_2$ quotients of X_{22}. Since Enriques surfaces are projective and have $h^{0,2}(E) = h^{2,0}(E) = 0$, we have the following corollary of Theorem 1.2:

Corollary 1.6. *Let $(M^5, \mathcal{S})$ be a regular Sasakian structure over an Enriques surface. Then $(M^5, \mathcal{S})$ is $\mathcal{S}$-stable.*

Remark 1.7. One can consider Enriques surfaces as K3 orbifolds with a trivial orbifold structure. Its canonical bundle K_E is not trivial, but K_E^2 is. On the other hand K3 orbifolds of the form X_{22}/G where G is a finite group acting on X_{22} that leaves its holomorphic $(2,0)$ form invariant have $\pi_1^{orb}(X_{22}/G) = G$ and a trivial canonical bundle. They have been studied [Nik76, Fuj83] and classified by Mukai [Muk88]. As pointed out by Kollár [Kol05] the best known example where $\pi_1^{orb}(X) \neq \mathbb{1}$ is the well known Kummer surface $X = \mathbb{T}^2/\mathbb{Z}_2$ in which case $\pi_1^{orb}(X)$ is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}^4$. It would be interesting to determine the stability properties of these structures.

It is convenient to describe Sasakian and transverse Kählerian structures in categorical language, so we assume the reader has some familiarity with this language as well as the relationship between orbifolds and étale Lie groupoids. We refer to Chapter 4 of [BG08] for the fundamentals as well as standard references [ALR07, MM03, Moe02].

2. The Transverse Kähler Flow and Sasaki Groupoids

An oriented 1-dimensional foliation is called a *flow*, and we are interested in transverse Hermitian and transverse Kähler flows. In particular they are *Riemannian flows*, and they form a special case of transverse Kähler foliations which were studied by El Kacimi Alaoui and collaborators [EKA90, EKAG97]. Their relation with Sasaki geometry was developed in [BG08] and developed further by Nozawa and collaborators [Noz14, GNT16]. We begin with the Riemannian foliations $\mathcal{F}$ (see [Mol88], chapter 2 of [BG08], and references therein) on a compact oriented manifold M and its basic cohomology ring $H^*_B(\mathcal{F})$. A Riemannian foliation $\mathcal{F}$ is said to be *homologically oriented* if $H^n_B(\mathcal{F})\neq 0$ where n is the (real) codimension of $\mathcal{F}$. If the foliation $\mathcal{F}$ is holomorphic with transverse complex structure $\bar{J}$ and has a compatible transverse Riemannian metric g^T such that $g^T\circ\bar{J}\otimes\mathbb{1}=\omega^T$ is a basic 2-form, the triple $(\mathcal{F},\bar{J},\omega^T)$ is called a **transverse Hermitian foliation**.

Note that for a Riemannian flow $\mathcal{F}$ a choice of Riemannian metric on M of the form $g=g^T+\eta\otimes\eta$, where η is the dual 1-form to a nowhere vanishing section ξ of $\mathcal{F}$, splits the exact sequence

$$0\longrightarrow\mathcal{F}\longrightarrow TM\longrightarrow TM/\mathcal{F}\longrightarrow 0$$

as

$$TM=\mathcal{F}\oplus\mathcal{D}, \tag{2}$$

and we have identified ω^T with a 2-form on $\mathcal{D}$, also denoted ω^T. If the transverse flow $\mathcal{F}$ is also homologically orientable, it follows from Molino and Sergiescu [MS85] that there exists a Riemannian metric g and a nowhere vanishing vector field ξ tangent to $\mathcal{F}$ such that ξ is a Killing vector field with respect to g. In this case the flow $\mathcal{F}$ is said to be **isometric** and the pair (g,ξ) is called a **Killing pair** in [Noz14]. Furthermore, the orbits generated by the Killing field ξ are geodesics of g. Without loss of generality we can take ξ to be a unit vector field, in which case we see that its dual *characteristic* 1-form η satisfies

$$\eta(\xi)=1,\qquad \xi\,\lrcorner\, d\eta=0.$$

This implies that $d\eta$ is basic and its basic cohomology class $[d\eta]_B$, called the **basic Euler class** of the isometric flow $\mathcal{F}$ by Saralegui [Sar85], is, up to multiplication, an invariant of the foliation. Note that $d\eta$ depends on the metric g, but its vanishing does not. A transverse Hermitian flow is said to be **trivial** if its basic Euler class vanishes in which case $(M, \mathcal{F})$ is a foliated bundle [Sar85]. For example $[d\eta]_B = 0$ if a finite cover of M is diffeomorphic to the product $N \times S^1$. We deal almost exclusively with nontrivial transverse Hermitian flows, that is we assume that M admits a one dimensional flow $\mathcal{F}$ with a non-zero Euler class, i.e. $[d\eta]_B \neq 0$. Note that if $b_1(M) = 0$, every transverse Hermitian flow on M is nontrivial. We only consider nontrivial isometric transverse Hermitian flows which we write as the quadruple (ξ, η, Φ, g) or $(\xi, \eta, \Phi, \omega^T)$ depending on the emphasis where

$$g = \omega^T \circ (\mathbb{1} \otimes J) + \eta \otimes \eta \tag{3}$$

with the endomorphism Φ is

$$\Phi = J \oplus (\mathbb{1} - \xi \otimes \eta) \tag{4}$$

and J is the complex structure on $\mathcal{D}$. The following equations hold:

$$\pounds_\xi \omega^T = 0, \quad \pounds_\xi \Phi = 0, \quad \Phi^2 = \begin{cases} J^2 = -\mathbb{1} & \text{on } \mathcal{D}, \\ 0 & \text{on } \mathcal{F}. \end{cases} \tag{5}$$

We emphasize here that the quadruple (ξ, η, Φ, g) is not necessarily a contact metric structure since ω^T is not necessarily $d\eta$. We also see

Lemma 2.1. *The pair $(\mathcal{D}, J)$ defines a CR structure on M*

2.1. **Transverse Kähler flows.** It is well known [BG08, Noz14] that the characteristic Reeb foliation $\mathcal{F}_\xi$ of a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a transverse Kähler flow with a nontrivial Euler class, i.e. $[d\eta]_B \neq 0$. However, the converse does not generally hold and one of the goals of this paper is to describe their relationship.

Definition 2.2. *A transverse Hermitian flow $(\mathcal{F}, J, \omega^T)$ is said to be a* **transverse Kähler flow** *if the basic 2-form ω^T is closed.*

We have

Lemma 2.3. *A compact oriented manifold M with a transverse Kähler flow $(\mathcal{F}, J, \omega^T)$ is homologically oriented.*

Proof. Since a flow is oriented and the basic cohomology class $[\omega^T]^p$ is nondegenerate in $H_B^{2p}(\mathcal{F}) \otimes \mathbb{R}$ for all $p = 1, \ldots, n$, a transverse Kähler flow is homologically oriented. □

From Lemma 2.3 and the preceeding discussion, we can represent a transverse Kähler flow by the quadruple $(\xi, \eta, \Phi, \omega^T)$ or (ξ, η, Φ, g).

However, such a quadruple is not unique as shown in Lemma 2.15 below.

As emphasized by El Kacimi-Alaou [EKA90] transverse Kähler foliations on compact manifolds possess the same properties as Kähler structures on compact manifolds, the Hodge decomposition, Lefschetz decomposition, etc. In particular, a transverse Kähler flow (ξ, η, Φ, g) gives rise to a basic Hodge decomposition (over $\mathbb{C}$),

$$H^n_B(\mathcal{F}) = \bigoplus_{p+q=n} H^{p,q}_B(\mathcal{F}), \qquad H^{q,p}_B(\mathcal{F}) = \overline{H^{p,q}_B(\mathcal{F})}, \tag{6}$$

which gives the basic Hodge numbers

$$h^{p,q}_B = \dim_{\mathbb{C}} H^{p,q}_B(\mathcal{F}). \tag{7}$$

We let $b^1_B = \dim H^1(M, \mathbb{R})$ denote the basic first Betti number. Then we easily see

Lemma 2.4. *Let $(M, \mathcal{F})$ be a compact transverse Kähler flow. Then the natural map $H^1_B(\mathcal{F}) \longrightarrow H^1(M, \mathbb{C})$ is injective, and $b^1_B = 2h^{1,0}_B = 2h^{0,1}_B$. In particular, $b_1(M) = 0$ implies $h^{1,0}_B = h^{0,1}_B = 0$.*

Definition 2.5. *We define the* **transverse Kähler cone** *$K^T(M, \mathcal{F})$ to be the set of all transverse Kähler classes $[\omega^T]_B$ in $H^{1,1}_B(\mathcal{F}) \cap H^2_B(\mathcal{F}) \otimes \mathbb{R}$.*

From the standard definition of Sasakian structure one sees

Lemma 2.6. *A transverse Kähler flow $(\xi, \eta, \Phi, \omega^T)$ is Sasakian if and only if $\omega^T = d\eta$ where η is a contact 1-form.*

Proof. The only if part is well known. Suppose that $\omega^T = d\eta$ which implies that $d\eta$ is type $(1,1)$. The transverse form $(\omega^T)^n_B$ is a basic volume form, so $\eta \wedge (\omega^T)^n$ is nowhere vanishing. Thus, η is a contact 1-form on M. So the quadruple (ξ, η, Φ, g) is a contact metric structure where Equations (3) and (4) hold, which implies that (ξ, η, Φ, g) is Sasakian. □

2.2. **Transverse Holonomy.** There are two distinct notions of transverse holonomy both of which are important to us. There is Haefliger's transverse holonomy groupoid, and for Riemannian foliations there is the holonomy group of the transverse Levi-Civita connection. It is the latter that concerns us here. We refer to Chapter 2 of [Joy07] for the description of holonomy groups on vector bundles. We can define the transverse Riemannian holonomy representation of a Riemannian foliation.

Definition 2.7. *For Riemannian foliations* $\mathcal{F}$ *the* **transverse holonomy group** $\mathrm{Hol}(\mathcal{F})$ *is the Riemannian holonomy group of the transverse Levi-Civita connection* ∇^T, *and* $\mathrm{Hol}^0(\mathcal{F})$ *denotes the restricted transverse holonomy group*[1].

In the case of transverse Kähler structures, the transverse complex structure is also parallel, i.e. $\nabla^T\bar{J} = 0$ and equivalently, $\nabla^T J = 0$ using the isomorphism defined by the splitting (2). It follows that

Lemma 2.8. *Let* M *be a compact manifold of dimension* $2n+1$ *with a transverse Hermitian flow* $(\mathcal{F}_\xi, \bar{J}, \omega^T)$. *Then the holonomy representation* $\mathrm{Hol}(\mathcal{F}_\xi, \bar{J}, \omega^T)$ *on* $TM/\mathcal{F}_\xi$ *lies in* $U(n) \subset GL(n, \mathbb{C})$ *if and only if* $(\mathcal{F}_\xi, \bar{J}, \omega^T)$ *is Kähler. Moreover, isomorphism*

$$(TM/\mathcal{F}_\xi, \bar{J}, \omega^T) \xrightarrow{\psi} (\mathcal{D}, J, \omega^T)$$

induced by the splitting (2) *induces an isomorphism of holonomy representations.*

Proof. Since the holonomy representation is defined only up to conjugation in $GL(n, \mathbb{C})$, the isomorphism ψ implies that holonomy representions of $\mathrm{Hol}(\mathcal{F})$ on $TM/\mathcal{F}_\xi$ and $\mathcal{D}$ are represented by conjugate subgroups of $GL(n, \mathbb{C})$. Moreover, $\mathrm{Hol}(\mathcal{F}) \subset \mathrm{U(n)}$ if and only if $\nabla^T\omega^T = 0$, $\nabla^T\bar{J} = 0$ if and only if ω^T is a basic closed 2-form if and only if the transverse Hermitian structure $(\mathcal{F}, \bar{J}, \omega^T)$ is Kähler. □

Equivalently, we state this with respect to the splitting (2).

Lemma 2.9. *Let* M *be a compact manifold with an isometric transverse Hermitian flow* (ξ, η, Φ, g). *Then* $\mathrm{Hol}(\mathcal{F}) \subset \mathrm{U(n)}$ *if and only if* (ξ, η, Φ, g) *is Kähler.*

The irreducible holonomy groups that are proper subgroups of $U(n)$ are $SU(n)$ and $Sp(\frac{n}{2})$ where the later occurs only for n even. In this paper we are interested in transverse Kähler flows $(\mathcal{F}, \bar{J}, \omega^T)$ whose transverse holonomy groups are either $SU(n)$ (transverse Calabi-Yau) or $Sp(n)$ (transverse hyperkähler).

2.3. **The Invariant Torus and its Invariant Cone.** For a homologically oriented Riemannian flow on a compact manifold there is a torus of isometries as described by the work of Molino and his coworkers [Mol79, Mol82, MS85, Mol88, Car84] which we now describe. The invariant cone should be viewed as a generalization of the Sasaki cone

[1]Recall that the restricted holonomy group is obtained by restricting the holonomy computation to null-homotopic loops. $\mathrm{Hol}^0(\mathcal{F})$ is the connected component of $\mathrm{Hol}(\mathcal{F})$.

[BGS08]. For transverse Kähler flows we call this the *invariant cone*, keeping in mind that if the transverse Kähler flow is Sasakian it coincides with the Sasaki cone.

Applying the basic Hodge decomposition (6) to the basic Euler class gives

$$[d\eta]_B = [d\eta^{2,0}]_B + [d\eta^{1,1}]_B + [d\eta^{0,2}]_B, \qquad d\eta^{0,2} = \overline{d\eta^{2,0}} \tag{8}$$

and as we shall see below $d\eta^{2,0}$ (equivalently $d\eta^{0,2}$) is an obstruction for the transverse Kähler flow to be Sasakian. We have

Proposition 2.10. *Let (ξ, η, Φ, g) be a transverse Kähler flow on a compact manifold M of dimension $2n+1$. Then there exists a k-dimensional Abelian Lie algebra $\mathfrak{a}(M, \mathcal{F})$ of Killing vector fields with $1 \leq k \leq n+1$ that is independent of the transverse Kähler metric and commutes with all transverse vector fields. Moreover, $\xi \in \mathfrak{a}(M, \mathcal{F})$.*

Proof. This follows from a result of Molino (see Theorem 5.2 in [Mol88]) that says that on any compact manifold with a Riemannian foliation there exists a locally constant sheaf of germs $C(M, \mathcal{F})$ of locally transverse commuting Killing fields such that

(1) all global transverse vector fields commute with $C(M, \mathcal{F})$,
(2) $C(M, \mathcal{F})$ is independent of the transverse metric g^T.

$C(M, \mathcal{F})$ is called the *commuting sheaf* in [Mol88] and the *faisceau transverse central* in [Mol79, Mol82], and it is an invariant of the foliation $\mathcal{F}$. We apply this to the case that M has a transverse Kähler flow $(\mathcal{F}, \bar{J}, \omega^T)$. In this case $\mathcal{F}$ is homologically oriented, so by [MS85] the sheaf $C(M, \mathcal{F})$ has a global trivialization. This gives an Abelian Lie algebra $\mathfrak{a}^T(M, \mathcal{F})$ of global transverse vector fields associated to $\mathcal{F}$ that is independent of the transverse metric and commutes with all transverse vector fields. Thus, there is a nowhere vanishing smooth vector field ξ tangent to $\mathcal{F}$ that commutes with all transverse vector fields. This implies that $\mathfrak{a}^T(M, \mathcal{F})$ extends to a k-dimensional Abelian Lie algebra

$$\mathfrak{a}(M, \mathcal{F}_\xi) = \mathfrak{a}^T(M, \mathcal{F}) \oplus \mathbb{R}\xi \tag{9}$$

with $1 \leq k \leq n+1$ which is independent of the transverse Kähler metric and commutes with all transverse vector fields. Moreover, since the elements of $\mathfrak{a}^T(M, \mathcal{F})$ are Killing fields with respect to any transverse Kähler metric and transverse Kähler forms are harmonic with respect to the basic Laplacian Δ_B, the elements of $\mathfrak{a}(M, \mathcal{F})$ also leave $\bar{J}$ invariant. Clearly, by construction $\xi \in \mathfrak{a}(M, \mathcal{F}_\xi)$. □

We now define the transverse Kähler analogue of the Sasaki cone, namely the **invariant cone**

$$\mathfrak{a}^+(M,\mathcal{F}_\xi)=\{\xi'\in\mathfrak{a}(M,\mathcal{F}_\xi)\mid \eta(\xi')>0\} \tag{10}$$

where η is the dual 1-form to ξ. Such elements take the form

$$\xi'=\bar{X}+\eta(\xi')\xi \tag{11}$$

where $\bar{X}\in\mathfrak{a}^T(M,\mathcal{F}_\xi)$. The function $\eta(\xi')$ is basic with respect to $\mathcal{F}_\xi$ and is called a **Killing potential**.

Lemma 2.11. *The set $\mathfrak{a}^+(M,\mathcal{F}_\xi)$ is a convex cone. Moreover, for any $\xi'\in\mathfrak{a}^+(M,\mathcal{F}_\xi)$ we have $\mathfrak{a}^+(M,\mathcal{F}_{\xi'})=\mathfrak{a}^+(M,\mathcal{F}_\xi)$.*

Proof. Consider the transverse homothety defined by $\xi\mapsto a^{-1}\xi,\ \eta\mapsto a\eta$ and $\omega^T\mapsto a\omega^T$. Then from (3) we see that if ξ is in $\mathfrak{a}^+(M,\mathcal{F}_\xi)$, then so is $(g_a,a^{-1}\xi)$ where

$$g_a=a\omega^T\circ(\mathbb{1}\oplus\bar{J})+a^2\eta\otimes\eta.$$

So $\mathfrak{a}^+(M,\mathcal{F}_\xi)$ is a cone. Now suppose $\xi_0,\xi_1\in\mathfrak{a}^+(M,\mathcal{F}_\xi)$, and consider the line segment $\xi_t=\xi_0+t(\xi_1-\xi_0)$ then

$$\eta_0(\xi_t)=(1-t)+t\eta_0(\xi_1)>0$$

for all $t\in[0,1]$ implying that $\mathfrak{a}^+(M,\mathcal{F}_\xi)$ is convex. To prove the last statement we note that if $\xi'\in\mathfrak{a}(M,\mathcal{F}_\xi)$, than $\mathfrak{a}(M,\mathcal{F}_\xi)$ and $\mathfrak{a}(M,\mathcal{F}'_\xi)$ are isomorphic Abelian Lie algebras. So we only need to prove that $\xi\in\mathfrak{a}^+(M,\mathcal{F}_{\xi'})$. But this follows by construction since $\eta'(\xi)=(\eta(\xi'))^{-1}>0$. □

Next as in Lemma 2 of [AC18] we have:

Lemma 2.12. *Suppose $\xi_0,\xi_1\in\mathfrak{a}^+(M,\mathcal{F}_\xi)$ then $\xi_1=\eta_0(\xi_1)\xi_0 \mod \ker\eta_0$.*

Remark 2.13. In the Sasaki category fixing a Sasakian structure $\mathcal{S}_0=(\xi_0,\eta_0,\Phi_0,g_0)$ and a nowhere vanishing smooth function f the vector field $f\xi_0$ defines a *weighted Sasakian structure* in the sense of [AC18] when f is chosen to be a nowhere vanishing Killing potential $\eta_0(\xi_1)$ with respect to $\mathcal{S}_0$.

A result of Carrière [Car84] states that the leaf closure $\overline{\mathcal{F}}$ of a Riemannian flow is diffeomorphic to a real torus $\mathbb{T}$ of real dimension k and that $\mathcal{F}$ restricted to a leaf is conjugate to a linear flow on $\mathbb{T}$. Clearly, in any case $\mathbb{T}$ is contained in a maximal torus $\mathbb{T}_{max}$. In the case of a transverse Kähler flow of real dimension $2n+1$ we have the range $1\leq k\leq n+1$ for the dimension k of $\mathbb{T}$. The dimension k is an invariant of the flow called its *toral rank*. Hence, associated to each quadruple (ξ,η,Φ,ω^T) is a maximal torus $\mathbb{T}^k$ that leaves (ξ,η,Φ,ω^T) invariant.

$\mathbb{T}^k$ is called the **invariant torus** of the isometric transverse Kähler flow (ξ,η,Φ,ω^T).

Proposition 2.14. *The invariant torus $\mathbb{T}^k$ is independent of the transverse Kähler metric ω^T and the choice of Reeb field in $\mathfrak{a}^+(M,\mathcal{F})$. Hence, it is independent of the pair $(\xi,[\omega^T]_B)\in\mathfrak{a}^+(M,\mathcal{F})\times K^T(M,\mathcal{F})$.*

If we fix a transverse Kähler flow $(\xi_o,\eta_o,\Phi_o,g_o)$ we obtain a family of transverse Kähler flows associated to $(\xi_o,\eta_o,\Phi_o,g_o)$, namely the disjoint union

$$\bigsqcup_{\xi\in\mathfrak{a}^+(M,\mathcal{F}_o)} K^T(M,\mathcal{F}_\xi)=\{(\xi,\eta,\Phi,\omega^T)\mid \xi\in\mathfrak{a}^+(M,\mathcal{F}_{\xi_o}),\ [\omega^T]_B\in K^T(M,\mathcal{F}_{\xi_o})\}$$

which is isomorphic to the diagonal in the product $\mathfrak{a}^+(M,\mathcal{F}_{\xi_1})\times K^T(M,\mathcal{F}_{\xi_2})$.

As in the Sasaki case, Section 7.5.1 of [BG08], we give this family the C^∞ compact-open topology as sections of vector bundles. This gives a smooth family of transverse Kähler flows within a fixed basic cohomology class $[\omega^T]_B$, and as in Section 6 of [BGS08] we obtain a smooth family when fixing the underlying CR structure and varying $\xi'\in\mathfrak{a}^+(M,\mathcal{F})$ which implies that the family $\mathcal{S}(\mathcal{F},\bar{J})$ is smooth.

Lemma 2.15. *Let (g,ξ) and (g',ξ) be two Killing pairs associated to the transverse Kähler flow $(\mathcal{F}_\xi,\bar{J},\omega^T)$ with the same basic Euler class. Then there exists a basic 1-form ζ such that*

$$g'=g+\zeta\otimes\eta\oplus\eta\otimes\zeta\oplus\zeta\otimes\zeta.$$

Proof. The dual 1-forms η,η' satisfy $g(\xi,X)=\eta(X)$ and $g'(\xi,X)=\eta'(X)$, and since $[d\eta']_B=[d\eta]_B$ there exists a basic 1-form ζ such that $d\eta'=d\eta+d\zeta$. But g is given by Equation (3) and

$$g'=\omega^T\circ(\mathbb{1}\otimes J)\oplus\eta'\otimes\eta'$$

which gives the result. □

Remark 2.16. Note that fixing the CR structure fixes the Killing pair (g,ξ).

Remark 2.17. The contact 1-form η in a quasiregular Sasakian structure can be viewed as a connection in a principal S^1 orbibundle over a projective algebraic orbifold. Two such connections forms η,η' are said to be **gauge equivalent** if there exists a smooth basic function f such that $\eta'=\eta+df$. One easily sees that such gauge transformed contact metric structures of a Sasakian structure are all Sasakian. This gives rise to gauge equivalences classes of Sasakian structure. Moreover, gauge equivalent Sasakian structures have the same underlying transverse Kähler flow, and a choice of Killing pair uniquely determines the gauge together with the class $[\zeta]_B$ in $H^1_B(\mathcal{F})_{\mathbb{R}} \approx H^1(M,\mathbb{R})$.

2.4. **The Groupoids.** For the categorical approach we begin with the category $\mathfrak{S}$ whose objects are smooth oriented compact connected manifolds and whose morphisms are smooth maps. Actually, we shall work in certain categories over $\mathfrak{S}$. We define the category $\mathcal{KTK}$ of *transverse Kähler flows* whose objects are the compact oriented and homologically oriented manifolds M together with a transverse Kähler flow $(\mathcal{F},\bar{J},\omega^T)$ and whose morphisms are smooth maps $\psi: M\longrightarrow M'$ that intertwine the triples $(\mathcal{F},\bar{J},\omega^T)$, that is

$$\mathcal{F}'=\psi_*\mathcal{F},\quad \psi_*\circ\bar{J}=\bar{J}'\circ\psi_*,\quad \psi^*\omega'^T=\omega^T. \tag{12}$$

whenever this makes sense. $p:\mathcal{KTK}\longrightarrow\mathfrak{S}$ is a category fibered in groupoids $\mathcal{G}TK$ over $\mathfrak{S}$. For an object M of $\mathfrak{S}$ the groupoid is the category $\mathcal{KTK}(M)$ whose objects are transverse Kähler flows $(\mathcal{F},\bar{J},\omega^T)$ with ω^T a basic positive definite 2-form on M, and whose morphisms are diffeomorphisms. Similarly we have the category of *isometric transverse Kähler structures* $\mathcal{I}KT$ whose objects are the the quadruples (ξ,η,Φ,g) described above, and whose morphisms are smooth maps that satisfy

$$\psi_*\xi=\xi',\qquad \psi^*\eta'=\eta,\qquad \psi_*\circ\Phi=\Phi'\circ\psi_*,\qquad \psi^*g'=g \tag{13}$$

whenever this makes sense. $\mathcal{I}KT$ is also a category fibered in groupoids $\mathcal{G}\mathcal{I}TK$. By Lemma 2.6 we consider the category of Sasakian structures $\mathcal{SC}$ to be the subcategory of $\mathcal{KTK}$ whose objects satisfy $\omega^T=d\eta$ for some contact 1-form η. Of course, this is also a category fibered in groupoids $\mathcal{SG}$. The following lemma (see [Noz14] for a cogent proof) implies that the forgetful functor

$$F:\mathcal{SG}\longrightarrow\mathcal{G}TK$$

factors through the groupoid $\mathcal{G}\mathcal{I}TK$. However, this factorization is not unique.

Lemma 2.18. *A transverse Kähler flow $(\mathcal{F},\bar{J},\omega^T)$ is the flow of a Sasakian structure (ξ,η,Φ,g) if and only if η is a contact 1-form, ξ its Reeb vector field such that $\mathcal{F}_\xi=\mathcal{F}$ and $\omega^T=d\eta$. Moreover, two Sasakian structures (ξ,η,Φ,g) and (ξ,η',Φ',g') correspond to the same transverse Kähler flow if there exists a closed basic 1-form ζ such that $\eta'=\eta+\zeta$.*

Since we work exclusively on compact manifolds M, we are most interested in the subgroupoids $\mathcal{S}_1\mathcal{G}$ and $\mathcal{G}_1\mathcal{I}TK$ consisting of those structures such that the Killing vector field generates a locally free S^1 action on M. We work almost entirely with these subgroupoids viewing the full groupoids as completions in the sense of Carriére as described below.

Then as discussed above this is equivalent to the standard description of Sasakian structures $\mathcal{S} = (\xi, \eta, \Phi, g)$ defining the category $\mathcal{SM}$ whose objects are pairs $(M, \mathcal{S})$ where M is an object of $\mathfrak{S}$ and $\mathcal{S}$ is Sasakian structure on M. Morphisms are smooth maps. As with $\mathcal{G}TK$ the category $\mathcal{SM}$ is a category fibered in groupoids $\mathcal{SG}$ where $\mathcal{SG}$ is the groupoid whose objects are pairs $(M, \mathcal{S})$ and morphisms are diffeomorphisms that *intertwine* the Sasakian structures, that is, diffeomorphisms that satisfy Equations (13). The **automorphism group** $\mathfrak{Aut}(\mathcal{F}, \bar{J}, \omega^T)$ of a transverse Kähler flow $(\mathcal{F}, \bar{J}, \omega^T)$ is by definition the isotropy group of the groupoid $\mathcal{G}TK$ at $(\mathcal{F}, \bar{J}, \omega^T)$. Since we have fixed the Riemannian metric g of Equation (3), $\mathfrak{Aut}(\mathcal{F}, \bar{J}, \omega^T)$ is a compact Lie group, and thus has a maximal torus $\mathbb{T}_{max}$ defined up to conjugacy. A result of Carrière [Car84] states that the leaf closure $\overline{\mathcal{F}}$ of a Riemannian flow is diffeomorphic to a real torus $\mathbb{T}$ of real dimension k and that $\mathcal{F}$ restricted to a leaf is conjugate to a linear flow on $\mathbb{T}$. Clearly, in any case $\mathbb{T}$ is contained in a maximal torus $\mathbb{T}_{max}$. In the case of a transverse Kähler flow of real dimension $2n+1$ we have the range $1 \leq k \leq n+1$ for the dimension k of $\mathbb{T}$. The dimension k is an invariant of the flow called its *toral rank*. This gives rise to a stratification of the groupoid $\mathcal{G}TK$ according to toral rank. The case of toral rank 1 is just the quasiregular case with a locally free S^1 action and the set of quasiregular transverse Kähler flows, which is dense in $\mathcal{G}TK$, is denoted by $\mathcal{G}_1TK$. So $\mathcal{G}_1TK$ is an embedded subgroupoid and $\mathcal{G}TK$ can be viewed as a completion of $\mathcal{G}_1TK$. Most of our considerations are about $\mathcal{G}_1TK$. As with $\mathcal{G}TK$, the Sasaki groupoid $\mathcal{SG}$ is stratified by its toral rank, and $\mathcal{S}_1\mathcal{G}$ is dense in $\mathcal{SG}$. Note that the functor F preserves toral rank. From Lemma 2.18 the fiber $F^{-1}(\mathcal{K})$ is either empty or the Abelian group $Z^1_B(M)$ of closed basic 1-forms on M. Elements $\zeta \in Z^1_B(M)$ are invariant under $\mathbb{T}$. From [MP97, Moe02, MM03] we have as detailed in the Appendix

Proposition 2.19. *$\mathcal{G}_1TK$ is a holomorphic foliation Lie groupoid with finite cyclic isotropy groups and is Morita equivalent to a proper étale Lie groupoid.*

This proposition gives an equivalence between the transverse geometry of quasiregular structures on M and the geometry on the quotient orbifold $X = M/S^1$.

Remark 2.20. Viewing foliations as holonomy groupoids makes the groupoid $\mathcal{G}TK$ into a 2-category whose objects are holonomy groupoids, whose 1-morphisms are functors between them, and 2-morphisms are morphisms between parallel functors.

We are mainly interested in the local behavior of the groupoid $\mathcal{G}TK$ near the transverse Kähler structure of a Sasakian structure. The groupoids $\mathcal{G}TK$ and $\mathcal{S}\mathcal{G}$ are generally not locally equivalent; nevertheless, many of the properties of Sasakian structures hold also for transverse Kähler flows. In particular, the basic cohomology complex holds for transverse Kähler flows, most notably the transverse Hodge decomposition and the basic first Chern class $c_1(\mathcal{F}_\xi) \in H^{1,1}_B(\mathcal{F}_\xi)$ [EKAH86, EKA90]. So one can talk about the *type* of a transverse Kähler flow as either positive, negative, null, or indefinite.

Proposition 2.21. *The forgetful functor*

$$F : \mathcal{S}_1\mathcal{G} \longrightarrow \mathcal{G}_1 TK$$

(1) *is a smooth functor of proper foliation Lie groupoids (orbifold groupoids) with finite cyclic isotropy groups;*
(2) *is faithful;*
(3) *is full if* $b_1(M) = 0$.

Proof. Both $\mathcal{S}_1\mathcal{G}$ and $\mathcal{G}_1TK$ are foliation Lie groupoids with finite isotropy groups by [Moe02] which are finite cyclic groups since the foliations are one dimensional. It is straightforward to check that F is smooth. A forgetful functor is always faithful, but not necessarily full nor essentially surjective on objects. Item (3) is due to Nozawa [Noz14]. We sketch his argument. We know that a non-empty fiber of F is the Abelian group $Z^1_B(M)$ of closed basic 1-forms.

Let $\mathcal{K} = (\mathcal{F}_\xi, \bar{J}, d\eta)$ be a transverse Kähler flow of Sasaki type, and consider the automorphism group $\mathfrak{Aut}(\mathcal{K})$ and we denote its connected component by $\mathfrak{Aut}_0(\mathcal{K})$. A morphism $\psi \in \mathcal{G}TK_{mor}$ sending $\mathcal{K}$ to $\mathcal{K}'$ induces an isomorphism of Lie groups $\mathfrak{Aut}_0(\mathcal{K}') \approx \mathfrak{Aut}_0(\mathcal{K})$. Define the *Hamiltonian* subgroup

$$\mathfrak{Ham}(\mathcal{K}) = \{\phi \in \mathfrak{Aut}_0(\mathcal{K}) \mid [\phi^*\eta - \eta] = 0\}.$$

$\mathfrak{Ham}(\mathcal{K})$ acts on the set of Sasakian structures $F^{-1}(\mathcal{K})$ and splits them into orbits. Nozawa identifies the orbit space $F^{-1}(\mathcal{K})/\mathfrak{Ham}(\mathcal{K})$ with the cohomology group $H^1(M, \mathbb{R})$. We need to show that F is surjective on morphisms. So we need to show that given Sasakian structures $\mathcal{S}$ and $\mathcal{S}'$ such that

$$F(M, \mathcal{S}') = (M, \mathcal{F}_{\xi'}, \bar{J}', d\eta'), \qquad F(M, \mathcal{S}) = (M, \mathcal{F}_\xi, \bar{J}, d\eta)$$

and a diffeomorphism $\psi : M \longrightarrow M$ satisfying Equation (12), there exists a diffeomorphism $\tilde{\psi} : M \longrightarrow M$ such that $\tilde{\psi}(\mathcal{S}) = \mathcal{S}'$ and $F(\tilde{\psi}) = \psi$.

If $b_1(M) = 0$ then $F^{-1}(\mathcal{K})$ consists of precisely one orbit, namely, the group $\mathfrak{Ham}(\mathcal{K})$. So there exists $\tilde{\psi} \in \mathfrak{Ham}(\mathcal{K})$ such that the diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\tilde{\psi}} & \mathcal{S}' \\ \Big\downarrow F & & \Big\downarrow F \\ \mathcal{K} & \xrightarrow{\psi} & \mathcal{K}' \end{array}$$

commutes implying that F is full in this case. □

Remark 2.22. When $H^1(M, \mathbb{R}) \neq 0$ there are many $\mathfrak{Ham}(\mathcal{K})$-orbits, so $\mathcal{S}$ and $\mathcal{S}'$ may lie on distinct orbits even though $F(\mathcal{S})$ and $F(\mathcal{S}')$ are isomorphic.

The next result shows that the functor F is never injective on objects.

Lemma 2.23. *Let $(M, \mathcal{S})$ be a Sasaki manifold with $\mathcal{S} = (\xi, \eta, \Phi, g)$. Then $\mathcal{S}_\zeta = (\xi, \eta + \zeta, \Phi_\zeta, g_\zeta)$is Sasakian for all $\zeta \in Z^1_B(M)$, where*

$$\Phi_\zeta = \Phi + \xi \otimes \zeta, \qquad g_\zeta = g + \zeta \otimes \eta + \eta \otimes \zeta + \zeta \otimes \zeta.$$

Moreover, $F(\mathcal{S}_\zeta) = F(\mathcal{S})$.

Proof. The first statement is proved in Section 7.5 of [BG08], and the second statement follows from the fact that $\Phi_\zeta \equiv \Phi \mod (\mathcal{F}_\xi)$. □

3. Deformation Theory of Transverse Kähler Flows

We begin by discussing the deformation theory of transverse holomorphic foliations. The well known Kodaira-Spencer deformation theory of complex manifolds has been completed by Kuranishi [Kur71] and applied to other pseudogroup structures [Kod60, KS61]. In particular the deformation theory of transverse holomorphic foliations has been studied extensively [DK79, DK80, GM80, GHS83]. See also Section 8.2.1 of [BG08]. Since the characteristic foliation $\mathcal{F}_\xi$ of a Sasakian structure is a transverse holomorphic foliation of dimension one, we can apply this theory to $\mathcal{F}_\xi$ when the manifold is compact. We can parameterize the transverse complex structures on a Sasaki manifold M by a complex analytic scheme[2] $(S, 0)^T$ that is the zero set of a finite number of holomorphic functions and a (not necessarily reduced) germ at 0. The main result is the following theorem of Girbau, Haefliger, and Sundararaman

[2]Schemes are needed here since the map Ψ in Theorem 3.1 vanishes to first order.

Theorem 3.1 ([GHS83]). *Let $\mathcal{F}$ be a transverse holomorphic foliation on a compact manifold M, and let $\Theta_{\mathcal{F}}$ denote the sheaf of germs of transversely holomorphic vector fields. Then*

1. *There is a germ $(S,0)^T$ of an analytic space (called the Kuranishi space) parameterizing a germ of a deformation $\mathcal{F}_s$ of $\mathcal{F}$ such that if $\mathcal{F}_{s'}$ is any germ of a deformation parameterizing $\mathcal{F}$ by the germ $(S',0)^T$, there is a holomorphic map $\phi:(S',0)^T\longrightarrow(S,0)^T$ so that the deformation $\mathcal{F}_{\phi(s)}$ is isomorphic to $\mathcal{F}_{s'}$.*
2. *The Kodaira-Spencer map $\rho: T_0S\longrightarrow H^1(M,\Theta_{\mathcal{F}})$ is an isomorphism.*
3. *There is an open neighborhood $U\subset H^1(M,\Theta_{\mathcal{F}})$ and a holomorphic map $\Psi: U\longrightarrow H^2(M,\Theta_{\mathcal{F}})$ such that $(S,0)^T$ is the germ at 0 of $\Psi^{-1}(0)$. The 2-jet of Ψ satisfies $j^2\Psi(u)=\frac{1}{2}[u,u]$.*

Here u is a 1-form with coefficients in the sheaf $\Theta_{\mathcal{F}}$, and the element $[u,u]\in H^2(M,\Theta_{\mathcal{F}})$ is the primary obstruction to performing the deformation. Item (i) of Theorem 3.1 says that the analytic space $(S,0)^T$ is versal. Moreover, if $H^2(M,\Theta_{\mathcal{F}})=0$ then a versal deformation exists and the Kuranishi space $(S,0)^T$ is isomorphic to an neighborhood of 0 in $H^1(M,\Theta_{\mathcal{F}})$. Note that as described in [GHS83] Kodaira-Spencer-Kuranishi deformation theory works equally well on compact complex orbifolds. In categorical language, this follows from Proposition 2.19.

Given this first order isomorphism, it is natural to ponder whether actual deformations exist and what their set of equivalences are, that is, describe the moduli space. However, here we restrict ourselves to paint a picture of the local moduli space, namely, the Kuranishi space of deformations of transverse holomorphic flows. We apply Theorem 3.1 to the case that the foliation $\mathcal{F}$ is also transversely Kähler with respect to the holomorphic structure. In this case El Kacimi Alaoui and Gmira have proven the following Stability Theorem (see also [EKA88] for the equivalent orbifold case):

Theorem 3.2 ([EKAG97]). *Let $\mathcal{F}_0$ be a homologically oriented transversely holomorphic foliation on a compact manifold M with a compatible transverse Kähler metric. Then there exists a neighborhood U of the germ $\mathcal{F}_0$ in the Kuranishi space S such that for all $t\in U$ the holomorphic foliation $\mathcal{F}_t$ has a compatible transverse Kähler metric.*

Using Lemma 2.3 we apply this theorem to the case where the foliation $\mathcal{F}$ has dimension one, that is to transverse Kähler flows:

Theorem 3.3. *Let $(\mathcal{F}_0,J,\omega^T)$ be a transverse Kähler flow on a compact oriented manifold M. Then there exists a neighborhood U of the germ $\mathcal{F}_0$ in the Kuranishi space S such that for all $t\in U$ the holomorphic flow*

$\mathcal{F}_t$ has a compatible transverse Kähler metric ω_t^T making $(\mathcal{F}_t, J_t, \omega_t^T)$ a transverse Kähler flow. In particular, the Kähler flows $(\mathcal{F}_t, J_t, \omega_t^T)$ are all isometric.

We are ready for

Definition 3.4. *Let $(M, \mathcal{S}_0)$ be a Sasaki manifold. We say that $\mathcal{S}_0$ is* **$\mathcal{S}$-stable** *if there exists a neighborhood N of $(\mathcal{F}_0, \bar{J}_0)$ in the Kuranishi space such that $(\mathcal{F}_t, \bar{J}_t)$ is Sasakian for all $t \in N$.*

Goertsches, Nozawa, and Töben proved that the basic Hodge numbers of a compact Sasaki manifold depend only on the underlying CR structure, Theorem 4.5 of [GNT16]. The question arises as to whether the analogue of this holds for a general transverse Kähler manifold. Generally, we do not know; however, we do have what we need, namely

Lemma 3.5. *There exists a neighborhood $N \subset S$ of the transverse Kähler flow $(\mathcal{F}_0, J_0, \omega_0^T)$ in the Kuranishi space S such that $h^{p,q}(\mathcal{F}_t, \bar{J}_t) = h^{p,q}(\mathcal{F}_0, \bar{J}_0)$ for $p+q = 2$ and for all $t \in N$. Furthermore, the holomorphic foliation $(\mathcal{F}_t, \bar{J}_t)$ has a compatible transverse Kähler form ω_t^T.*

Proof. We outline the proof following [EKAG97] which in turn followed [KS60]. The Lie groupoids $\mathcal{GS}$ and $\mathcal{G}TK$ have the Fréchet topology, and $\mathcal{G}TK$ has a smooth family of transversely strongly elliptic self adjoint 4th order differential operators

$$A_t = \partial_t\bar{\partial}_t\bar{\partial}_t^*\partial_t^* + \bar{\partial}_t^*\partial_t^*\partial_t\bar{\partial}_t + \bar{\partial}_t^*\partial_t\partial_t^*\bar{\partial}_t + \bar{\partial}_t^*\bar{\partial}_t + \partial_t^*\partial_t \tag{14}$$

acting on space of smooth basic (p,q)-forms. The kernel of A_t denoted by $\mathbf{F}_t^{p,q}$ is given by

$$\mathbf{F}_t^{p,q} = \{\alpha \in \Omega_B^{p,q}(\mathcal{F}) \mid \partial_t\alpha = 0, \quad \bar{\partial}_t\alpha = 0, \quad \bar{\partial}_t^*\partial_t^*\alpha = 0\}, \tag{15}$$

and we have the following orthogonal decomposition of smooth closed basic (p,q) forms

$$Z_B^{p,q}(\mathcal{F}) = \mathrm{im}(\partial_t\bar{\partial}_t) \oplus \mathbf{F}_t^{p,q}. \tag{16}$$

So the cohomology groups $H_B^{p,q}(\mathcal{F})$ are represented by elements of $\mathbf{F}_t^{p,q}$. Thus, by Proposition 6.3 of [EKAG97] there is a neighborhood N of the central fiber $(\mathcal{F}_0, J_0, \omega_0^T)$ such that for all $t \in N$ the dimension of $\mathbf{F}_t^{1,1}$ equals $h_B^{1,1}(\mathcal{F}_{\xi_t}, \bar{J}_t)$ and is independent of t, so $h_B^{1,1}(\mathcal{F}_{\xi_t}, \bar{J}_t) = h_B^{1,1}(\mathcal{F}_0, \bar{J}_0)$ in N. But since the basic 2nd Betti number b_2^B is independent of t and we have

$$b_B^2 = h^{2,0}(\mathcal{F}_{\xi_t}, \bar{J}_t) + h_B^{1,1}(\mathcal{F}_{\xi_t}, \bar{J}_t) + h_B^{0,2}(\mathcal{F}_{\xi_t}, \bar{J}_t) = h^{1,1}(\mathcal{F}_0, \bar{J}_0) + 2h_B^{2,0}(\mathcal{F}_{\xi_t}, \bar{J}_t)$$

which implies that $h_B^{2,0}(\mathcal{F}_{\xi_t}, \bar{J}_t)$ and $h_B^{0,2}(\mathcal{F}_{\xi_t}, \bar{J}_t) = \overline{h_B^{2,0}(\mathcal{F}_{\xi_t}, \bar{J}_t)}$ are also independent of t for all $t \in N$ which proves the first result. The second result also follows by Theorem 6.4 of [EKAG97]. □

Since $h_B^{p,q}(\mathcal{F}_\xi, J)$ are integer valued and $\mathfrak{a}^+(M,\mathcal{F}_\xi)$ is path connected, Lemma 3.5 implies

Proposition 3.6. *Let $(M,\mathcal{F}_\xi)$ be a compact transverse Kähler flow. Then $h_B^{p,q}(\mathcal{F}_{\xi'})$ is independent of $\xi' \in \mathfrak{a}^+(M,\mathcal{F}_\xi)$ for $p, q \leq 2$.*

This proposition together with the fact that $\mathcal{G}_1TK$ is dense in $\mathcal{G}TK$ allows us to reduce our arguments to the quasiregular case. Given this we shall often use Proposition 2.19 to pass between the transverse geometry of $(M,\mathcal{S})$ and the algebraic orbifold geometry of the quotient.

3.1. **An Obstruction to $\mathcal{S}$-stability.** We now consider obstructions to the stability of deformations of the transverse holomorphic foliations $(\mathcal{F}_\xi, J)$.

Lemma 3.7. *Let $(\mathcal{F}_0, \bar{J}_0, d\eta_0)$ be the transverse Kähler flow of a Sasakian structure $\mathcal{S}_0 = (\xi_0, \eta_0, \Phi_0, g_0)$. Under the deformation $(\mathcal{F}_0, \bar{J}_0, d\eta_0) \mapsto (\mathcal{F}_t, \bar{J}_t, d\eta)$, there is a neighborhood N of $(\mathcal{F}_0, \bar{J}_0, d\eta_0)$ in the Kuranishi space such that for all $t \in N$*

1. *the $(2,0)$ component of $d\eta$ is ∂-closed with respect to $\bar{J}_t$,*
2. *the $(0,2)$ component of $d\eta$ is $\bar{\partial}$-closed with respect to $\bar{J}_t$,*
3. *the $(1,1)$ component of $d\eta$ is Kähler with respect to $\bar{J}_t$ if and only if $d\eta^{2,0}$ is holomorphic and $d\eta^{0,2}$ is antiholomorphic,*
4. *$d\eta^{2,0} \wedge d\eta^{0,2} + (d\eta^{1,1})^2 > 0$,*
5. *$h^{p,q}(\mathcal{F}_t, \bar{J}_t) = h^{p,q}(\mathcal{F}_0, \bar{J}_0)$ for $p + q = 2$.*

Proof. The Hodge decomposition of the d_B-closed basic 2-form $d\eta$ with respect to the transverse holomorphic structure $(\mathcal{F}_\xi, J_t)$ is given by Equation (8). This shows that $d\eta^{2,0}$ is ∂-closed, $d\eta^{0,2}$ is $\bar{\partial}$-closed, and that

$$\bar{\partial} d\eta^{2,0} + \partial d\eta^{1,1} = 0, \qquad \bar{\partial} d\eta^{1,1} + \partial d\eta^{0,2} = 0.$$

So $d\eta^{1,1}$ will be closed if and only if $d\eta^{2,0}$ is holomorphic and $d\eta^{0,2}$ is antiholomorphic. Thus, in this case $d\eta^{1,1} \circ (\bar{J}_t \otimes \mathbb{1})$ will be a transverse Kähler metric in a neighborhood of the central fiber $(\mathcal{F}_\xi, \bar{J})$ which proves (1),(2), and (3). Item (4) follows from the Hodge decomposition and the fact that η is a contact 1-form. Item (5) holds by Lemma 3.5. □

Applying Lemma 3.7 to Sasaki manifolds shows that if $d\eta^{2,0}$ is a nonzero holomorphic section of $H^{2,0}(\mathcal{F}_\xi, \bar{J})$, we can deform to a transverse Kähler structure which is not necessarily associated to a Sasakian structure since $\omega^T = d\eta^{1,1} \neq d\eta$. Indeed, in Theorem 4.9 below we prove that this is the case for transverse hyperkähler structures. For any transverse holomorphic deformation of a Sasakian structure, we view the holomorphic section $d\eta^{2,0}$ as an obstruction to $\mathcal{S}$-stability[3].

3.2. **Proof of Theorem 1.2 and Corollary 1.3.** First we recall that by Proposition 2.21 the forgetful functor $F : \mathcal{S}\mathcal{G}(M,\mathcal{F}) \longrightarrow \mathcal{K}(M,\mathcal{F})^T$ is full and faithful.

Proof of Theorem 1.2. It remains to show that the vanishing of $h^{2,0}_B(\mathcal{S})$ and $h^{0,2}_B(\mathcal{S})$ implies the functor F is locally essentially surjective on objects. By Lemma 3.5 these Hodge numbers are independent of the Sasakian structure in $\mathfrak{a}^+(M,\mathcal{F}_\xi)$. So Lemma 3.7 implies that $h^{2,0}_B(\mathcal{S}_t) = 0$ for $t \in N_0$, a small enough neighborhood of the central fiber. So when we deform the transverse holomorphic foliation $\mathcal{F}, \bar{J}$, the basic Euler class $[d\eta]_B$ of $\mathcal{F}_t, \bar{J}_t$ must remain type $(1,1)$. It then follows from Theorem 1.1 of [Noz14] that the smooth family of flows $(\mathcal{F}_t, \bar{J}_t)$ are Sasakian in a possibly smaller neighborhood of $(\mathcal{F}, \bar{J})$. □

The proof of Corollary 1.3 now follows as in Proposition 7.1.7 of [Joy07].

3.3. **The Transverse Hodge Structure.** On Sasaki manifolds the basic cohomology has a transverse Hodge decomposition [EKA90] (see also Section 7.2.2 of [BG08]). An example of a weight 1 transverse Hodge structure and their deformations was studied by Nozawa [Noz14], where he considers Abelian varieties and their completions. Generally weight 1 transverse Hodge structures are much more complicated. For example, there are Sasaki 5-manifolds that are circle bundles over (orbifold) surfaces of general type with non-vanishing weight 1 transverse Hodge numbers.

Here we concern ourselves with weight 2 basic Hodge structures. However, as aptly noted in [GNT16] generally there is no Hodge structure in the usual sense, since a priori there is no integral lattice. Nevertheless, under the right circumstances we can construct an integral

[3]Nozawa identifies the $(0,2)$ component $(d\eta)^{0,2}$ as an obstruction to stability. Of course these are completely equivalent obstructions.

lattice. On a Sasaki manifold we have a diagram of exact sequences (17)

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & H^2(M,\mathbb{Z})\setminus Tor & & & & \\ & & & & \downarrow j & & & & \\ 0 \longrightarrow & H^0_B(\mathcal{F}_\xi) & \stackrel{\delta}{\longrightarrow} & H^2_B(\mathcal{F}_\xi) & \stackrel{\iota_*}{\longrightarrow} & H^2(M,\mathbb{R}) & \longrightarrow & H^1_B(\mathcal{F}_\xi) & \longrightarrow \cdots , \end{array}$$

where $\delta = \wedge[d\eta]_B$. So we can define transverse integral classes:

$$H^2_B(\mathcal{F}_\xi,\mathbb{Z}) = \{\beta \in H^2_B(\mathcal{F}_\xi) \mid \iota_*\beta = j(\alpha),\ \alpha \in H^2(M,\mathbb{Z})\setminus Tor\}. \tag{18}$$

Lemma 3.8. *Let $(M,\mathcal{S})$ be a Sasaki manifold with vanishing first Betti number $b_1(M)$. Then for each integral class $\alpha \in H^2(M,\mathbb{Z})\setminus Tor \subset H^2(M,\mathbb{R})$ there is a class $\beta \in H^2_B(\mathcal{F}_\xi)$ such that $\beta \equiv \alpha \mod \{[d\eta]_B\}$.*

Proof. Since for Sasaki manifolds $H^1_B(\mathcal{F}_\xi) \approx H^1(M,\mathbb{R})$ and the first Betti number $b_1(M)$ vanishes, ι_* is surjective. Moreover, we can construct a splitting of the short exact sequence over $\mathbb{Z}$ as follows: if α is an integral class as an element of $j(H^2(M,\mathbb{Z})/Tor) \subset H^2(M,\mathbb{R})$, there exists a $\beta \in H^2_B(\mathcal{F}_\xi)$ that equals the integral class α mod an element of the ideal generated by $[d\eta]_B$. □

We now apply this to the situation at hand in the case that $\mathcal{S}$ is quasiregular. Since α is an integral class we can consider β to be a 'transverse integral class'. Moreover, since $\mathcal{S}$ is quasiregular, the basic class $[d\eta]$ is the pullback of integral class in the orbifold cohomology of the base orbifold. So as such we can consider $[d\eta]_B$ to be an integral class in the basic cohomology. We denote the set of all such transverse integral classes by $H^2_B(\mathcal{F}_\xi,\mathbb{Z})$. This gives an exact sequence

$$0 \longrightarrow \mathbb{Z} \stackrel{\delta}{\longrightarrow} H^2_B(\mathcal{F}_\xi,\mathbb{Z}) \stackrel{\iota_*}{\longrightarrow} j\big(H^2(M,\mathbb{Z}\setminus \mathrm{Tor})\big) \longrightarrow 0$$

which splits by construction. We can now construct a 'transverse Hodge structure'.

Definition 3.9. *Let $(M,\mathcal{S})$ be a quasiregular Sasaki manifold with $b_1(M) = 0$. We say that $(M,\mathcal{S})$ has a* **transverse Hodge structure of weight** 2 *if there is a Hodge decomposition*

$$H^2_B(\mathcal{F}_\xi)\otimes\mathbb{C} = H^{2,0}_B(\mathcal{F}_\xi)\oplus H^{1,1}_B(\mathcal{F}_\xi)\oplus H^{0,2}_B(\mathcal{F}_\xi), \qquad H^{p,q}_B(\mathcal{F}_\xi) = \overline{H^{q,p}_B(\mathcal{F}_\xi)}$$

and an integral lattice Λ *such that* $\Lambda \otimes \mathbb{R} = H^2_B(\mathcal{F}_\xi)$. *We also define the* **transverse Neron-Severi group** *by*

$$NS(\mathcal{F}_\xi) = H^{1,1}_B(\mathcal{F}_\xi) \cap H^2_B(\mathcal{F}_\xi, \mathbb{Z})$$

and the **transverse Picard number** $n(\mathcal{F}_\xi)$ *to be its rank. The* **transverse Picard group** $\mathrm{Pic}^T(M, \mathcal{F}_\xi)$ *is the complex Abelian group (vector space)* $\Lambda^{\mathbb{C}} = (\Lambda \otimes \mathbb{C}) \cap H^{1,1}_B(\mathcal{F}_\xi)$ *of complex dimension* $n(\mathcal{F}_\xi)$. *The* **basic Hodge numbers** *are* $h^{p,q}_B(\mathcal{S}) = \dim H^{p,q}_B(\mathcal{F}_\xi)$.

Remark 3.10. Note that as defined the transverse Hodge structure is automatically polarized by the Reeb vector field ξ or equivalently by the transverse Kähler form $d\eta$ with a fixed underlying CR structure.

Proposition 3.11. *An isometric transverse Kähler flow manifold* $(M, \mathcal{K} = (\mathcal{F}_\xi, \bar{J}, \omega^T))$ *with* $b_1(M) = 0$ *admits a transverse Hodge structure of weight* 2.

Proof. Since $b_1(M) = 0$ Lemma 3.8 defines the notion of integral basic classes $H^2_B(\mathcal{F}_\xi, \mathbb{Z})$ and gives an isomorphism $H^2_B(\mathcal{F}_\xi, \mathbb{Z})/\{[d\eta]_B\} \approx H^2(M, \mathbb{Z})$. Since $\mathcal{S}$ is quasiregular, the class $[d\eta]_B$ is the image of an integral orbifold class $[\omega] \in H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. This then makes sense of $H^2_B(\mathcal{F}_\xi, \mathbb{Z})$ as a cohomology group over $\mathbb{Z}$ and provides an integral lattice $\Lambda = H^2_B(\mathcal{F}_\xi, \mathbb{Z})$. □

4. Transverse Kähler Holonomy

The irreducible transverse Kähler holonomy groups are

$$U(n), \qquad SU(n), \qquad Sp(n)$$

which correspond to irreducible transverse Kähler geometry, transverse Calabi-Yau geometry, and transverse hyperkähler geometry, respectively. Such general transverse structures were studied recently by Habib and Vezzoni [HV15]. They can be defined as holomorphic foliations whose transverse holonomy group is contained in $SU(n)$. Here we are interested in their relation with Sasakian geometry, so we specialize to the case of a holomorphic foliation of dimension one, namely the characteristic Reeb foliation $\mathcal{F}_\xi$. Sasaki manifolds with transverse holonomy contained in $SU(n)$ are null-Sasaki having vanishing transverse Ricci curvature by the transverse Yau Theorem [EKA90, BGM06]. They are called contact Calabi-Yau manifolds in [TV08].

Following Joyce [Joy07, GHJ03] we deal with irreducible transverse Calabi-Yau and irreducible transverse hyperkähler structures although we give the more general definitions below. Note that when $n = 2$ we have the equality $SU(2) = Sp(1)$, so Calabi-Yau and hyperkähler geometry coincide when $n = 2$. We note that the condition of irreducibility is crucial for the following stability results.

Remark 4.1. There is one other even dimensional irreducible Berger holonomy group that is related to Sasakian geometry, the group $Sp(n)\cdot Sp(1)$ whose transverse flows are twistor spaces of 3-Sasakian structures, cf. [BG99]; however, generally they are not Kähler and therefore, are not treated in this paper.

4.1. **Transverse Irreducible Calabi-Yau Structures.** Since $c_1(\mathcal{F}) = 0$ the transverse geometry is the geometry of compact Calabi-Yau orbifolds which has been studied in [Cam04] following the manifold case [Bog78, Bea83].

Definition 4.2. *We say that a transverse Kähler flow $(\mathcal{F}, \bar{J}, \omega^T)$ on a compact manifold M of dimension $2n+1$ is a* **transverse Calabi-Yau flow** *if its transverse holonomy group is contained in $SU(n)$. This transverse Calabi-Yau structure is* **irreducible** *if the transverse holonomy group equals $SU(n)$. We abbreviate irreducible transverse Calabi-Yau structures (flows) by* **ITCY**. *The ITCY flow is said to be of* **Sasaki type** *if $\omega^T = d\eta$ for some Sasakian structure (ξ, η, Φ, g).*

Calabi-Yau structures have holomorphic volume forms, so as expected transverse Calabi-Yau structures have transverse holomorphic volume forms, i.e holomorphic sections Ω^T of $H_B^{n,0}$. Since as mentioned above Calabi-Yau structures coincide with hyperkähler structures when $n = 2$, we assume in this section that $n > 2$.

We have following

Theorem 4.3. *Let M^{2n+1} be a compact manifold of dimension $2n+1$. If M admits a ITCY flow of Sasaki type $(\mathcal{F}, \bar{J}, d\eta)$ and $n > 2$ then $(\mathcal{F}, \bar{J}, d\eta)$ is $\mathcal{S}$-stable. Moreover, the Kuranishi space S is a open set in $H^1(M, \Theta)$.*

Proof. First we note, using the equivalence between transverse CY structures and CY orbifolds Proposition 2.19, that we can work on irreducible Calabi-Yau orbifolds X. Now since $n > 2$, as noted on page 125 of [Joy07], the induced action of the holonomy group $SU(n)$ on $\Lambda^{p,0}(X)$ fixes no complex $(p, 0)$ form for $0 < p < n$, and this implies that the Hodge numbers $h^{p,0}$ vanish in this range. The first statement is then an immediate corollary of Theorem 1.2.

The second statement is an orbifold version of a result of Tian [Tia87] which we now describe. So we let X be a compact Kähler orbifold and $\Gamma(X, \Omega^{p,q}(\Theta_X))$ be the set of global (p, q)-forms with coefficients in the sheaf of germs of holomorphic vector fields Θ_X or more generally for the sheaf of germs of any holomorphic tensor field. For the description of tensor fields on orbifolds we refer to [BGK05] as well as Section 4.4.2 of [BG08]. Generally, the canonical sheaf and the canonical orbisheaf are not equivalent; however, since transverse Calabi-Yau structures are null Sasakian structures, there are no branch divisors and Tian's proof is straightforward to generalize. From the GHS Theorem 3.1 we need to prove the existence of a one parameter family of solutions $\omega(t) \in \Gamma(X, \Omega^{0,1}(TX))$ with

$$\bar{\partial}\omega(t) + \frac{1}{2}[\omega(t), \omega(t)] = 0,\ \omega(0) = 0 \tag{19}$$

give a deformation of complex structures over X.

By the Taylor expansion (at the singular points, we consider the corresponding local covering spaces), we have $\omega(t) = w_1 t + w_2 t^2 + \cdots$ which we plug into (19). Given $\omega_1 \in \Gamma(X, \Omega^{0,1}(TX))$, we then need to solve the following system of equations inductively

$$\bar{\partial}\omega_N + \frac{1}{2}\sum_{i=1}^{N-1}[\omega_i, \omega_{N-i}] = 0,\ (N \geq 2). \tag{20}$$

Now we want to change (20) a bit. Since the canonical orbisheaf K_X is trivial, we have a natural isomorphism

$$i_q : \Gamma(X, \Omega^{0,q}(T_X)) \to \Gamma(X, \Omega^{n-1,q}).$$

For every $\Omega^{0,q}(T_X)$, locally we have

$$\phi = \sum_{\substack{i,J \\ |J|=q}} f^i_{\bar{J}} \frac{\partial}{\partial z^i} \otimes d\bar{z}^J,$$

and

$$i_q(\phi) = dz^1 \wedge \cdots \wedge dz^n(\phi).$$

It is easy to check that i_q is well-defined and isomorphic. Our goal is to replace ω_i in (20). To do that, we define

$$[i_1(\omega_1), i_1(\omega_2)] := i_2[\omega_1, \omega_2].$$

Thus given $\omega_1 \in \Gamma(X, \Omega^{n-1,1})$, we need to solve the following system of equations inductively

$$\bar{\partial}\omega_N + \frac{1}{2}\sum_{i=1}^{N-1}[\omega_i, \omega_{N-i}] = 0, \qquad \text{for } N \geq 2, \tag{21}$$

where $\omega_i \in \Gamma(X, \Omega^{n-1,1})$, $i = 2, 3, \ldots, N-1$. Since the proof of Lemma 3.1 of [Tia87] is local, it also holds in the orbifold case on the local uniformizing neighborhoods.

Lemma 4.4. *Let $\omega_1, \omega_2 \in \Gamma(X, \Omega^{n-1,1})$, then*

$$[\omega_1, \omega_2] = \partial(i^{-1}(\omega_1) \lrcorner \omega_2) - \#(\partial\omega_1) \wedge \omega_2 + \omega_1 \wedge \#(\partial\omega_2).$$

Then since the $\partial\bar{\partial}$-lemma for orbifold Hodge theory follows from the transverse version in [EKA90], the remainder of Tian's argument applies to our case. Indeed the proof of Theorem 1 of [Tia87] goes through verbatim. □

4.2. **Transverse Irreducible Hyperkähler Structures.** The seminal work on hyperkähler manifolds is [HKLR87]. Hyperkähler structures are a particular type of quaternionic structure to which we refer to Chapter 12 of [BG08], Chapter 10 of [Joy07], and Chapter 3 of [GHJ03] as well as the standard references [Huy99, Ver05].

Although we give the more general definition, henceforth by hyperkähler we shall mean the irreducible case, Hol $= Sp(n)$. We abbreviate irreducible transverse hyperkähler structures by **ITHK**.

Definition 4.5. *We say that a transverse Kähler flow $(\mathcal{F}, \bar{J}, \omega^T)$ on a compact manifold M of dimension $4n+1$ is a* **transverse hyperkähler flow** *if its transverse holonomy group is contained in $Sp(n)$. The transverse hyperkähler structure is* **irreducible** *if the transverse holonomy group equals $Sp(n)$.*

Remark 4.6. An equivalent definition of transverse hyperkähler is that the contact bundle $\mathcal{D}$ admits three almost complex structures $\{I_i\}_{i=1}^3$ that satisfy the algebra of the quaternions

$$I_i I_j = -\delta_{ij}\mathbb{1} + \epsilon_{ijk} I_k, \tag{22}$$

and the induced transverse antisymmetric forms $\omega_i^T = g \circ (I_i \otimes \mathbb{1})$ are covariantly constant ($\nabla^T \omega_i^T = 0$) with respect to the transverse Levi-Civita connection ∇^T.

It immediately follows from the definition that M has real dimension $4n+1$. Since we have an inclusion of holonomy groups $Sp(n) \subset SU(2n)$ a transverse hyperkähler structure is automatically a transverse null Kähler structure. We want to know when this transverse Kähler structure is Sasakian. There is a 1-1 correspondence between transverse hyperkähler flows and hyperkähler orbifolds, and these orbifolds are projective algebraic if and only if the canonical bundle $c_1(K_X) \in H^2_{orb}(X, \mathbb{Z})$.

Lemma 4.7. *Let (ξ, η, Φ, g) be a contact metric structure with a transverse hyperkähler structure $\{I_i\}_{i=1}^3$. Then fixing a transverse complex structure, say I_1, gives a null transverse Kähler structure ω_1^T. It is a Sasakian structure $\mathcal{S}_1 = (\xi, \eta, \Phi_1, g)$ with $\Phi_1 = I_1 \oplus \xi \otimes \eta$ if and only if $\omega^T = d\eta$. In this case we say that the hyperkähler structure is associated to the (necessarily null) Sasakian structure $\mathcal{S}_1$, or conversely the null Sasakian structure $\mathcal{S}_1$ is associated to the hyperkähler structure $\{I_i\}_{i=1}^3$.*

Proof. The condition $\nabla^T \omega_i = 0$ implies that ω_i are closed. But as in Lemma 2.2 of [Hit87] this implies that the transverse almost complex structures I_i are integrable and that the forms ω_i^T are Kähler with respect to the complex structure I_i. But clearly $\pounds_\xi \Phi_i = 0$ for $i = 1, 2, 3$ since I_i are endomorphisms of $\mathcal{D}$ and $\pounds_\xi \eta = 0$. This implies $\pounds_\xi \omega_i^T = 0$. Moreover, it is null, that is, $c_1(\mathcal{F}_\xi) = 0$ which implies $c_1(X)_{\mathbb{R}} = 0$ which in turn implies that $c_1(\mathcal{D})$ is a torsion class. □

Remark 4.8. As in the manifold case a transverse hyperkähler structure defines a transverse Kähler structure with a **transverse complex symplectic structure**. Explicitly, if (I_1, ω_1^T) defines the underlying transverse Kähler structure of the transverse hyperkähler structure, the complex 2-form $\omega_2^T + i\omega_3^T$ satifies $\omega_+^n \neq 0$ everwhere, and thus defines a transverse complex symplectic structure. Conversely, if we have a transverse Kähler structure (J, ω^T) together with a transverse holomorphic symplectic 2-form $\omega_{\mathbb{C}}^T$ that is covariantly constant with respect to the transverse Levi-Civita connnection ∇^T the conditions $\nabla^T \omega^T = \nabla^T J = \nabla^T \omega_{\mathbb{C}}^T = 0$ forces the transverse holonomy to lie in $Sp(n)$ where the real codimension of the foliation is $4n$ as in [Joy07] Section 10.4. This gives an equivalence between transverse hyperkähler structures and tranverse Kähler structures with a transverse complex symplectic structure.

Given a transverse hyperkähler structure we fix a transverse Kähler structure (I_1, ω_1^T) and its transverse complex symplectic structure $\omega_+ = \omega_2 + i\omega_3$. We now consider the proof of Theorem 1.4. First, we note that the 2nd statement in the theorem follows from [Cam04]. So it suffices to prove

Theorem 4.9. *Let $\mathcal{S}_1$ be an (ITHK) Sasaki structure on a compact manifold M with $b_1(M) = 0$ with the transverse Kähler structure defined by $I_1 \in \{I_i\}_{i=1}^3$. Then there are transverse Kähler deformations in the Kuranishi space $(S, 0)^T$ that are not Sasakian. So $\mathcal{S}_1$ is $\mathcal{S}$-unstable.*

Proof. Any null Sasakian structure $\mathcal{S}$ is quasiregular, so the quotient by the S^1 action generated by the Reeb vector field ξ is a Kähler polarized hyperkähler orbifold (X_1, ω_1) which represents the transverse hyperkähler structure. Letting the Sasakian structure $\mathcal{S}_1$ be the central fiber in the Kuranishi space S of transverse holomorphic deformations, there is a neighborhood $N \subset S$ of X_1 such that all $X_t \in N$ are transversely Kähler by Theorem 3.3. Now the transverse hyperkähler structure gives a 2-sphere's worth of transverse complex structures I_t defined by

$$I_t = t_1 I_1 + t_2 I_2 + t_3 I_3, \quad \text{with } t = (t_1, t_2, t_3) \text{ and } t_1^2 + t_2^2 + t_3^2 = 1. \tag{23}$$

We also have a 2-sphere's worth of transverse Kähler forms

$$\omega_t^T = \sum_{i=1}^{3} t_i \omega_i^T = \sum_{i=1}^{3} t_i g \circ (I_i \otimes 1\!\!1) = g \circ (I_t \otimes 1\!\!1), \qquad \sum_{i=1}^{3} t_i^2 = 1. \tag{24}$$

Since the transverse geometry is that of a Kähler orbifold, this gives rise to the twistor space $\mathcal{T}(X)$, a complex orbifold which is diffeomorphic as orbifolds to the product $X \times \mathbb{CP}^1$, but whose complex structure is not the product structure. It is more convenient to use stereographic coordinates $z \in \mathbb{C}$ defined by

$$t = (t_1, t_2, t_3) = \Big(\frac{1 - |z|^2}{1 + |z|^2}, -\frac{z + \bar{z}}{1 + |z|^2}, i \frac{z - \bar{z}}{1 + |z|^2} \Big) \tag{25}$$

with corresponding complex structure I_z. For each $z \in \mathbb{CP}^1$, there is an associated transverse Kähler structure. If we begin with a Sasakian structure with respect to $\Phi_0 = I_1 + \xi \otimes \eta$ and consider deformations of the transverse holomorphic structure leaving the transverse hyperkähler structure invariant, we obtain the complex structures I_z for $z \in \mathbb{CP}^1$. From this we get an induced complex structure on $\mathcal{T}(X)$ as follows. Using the natural projection $p : \mathcal{T}(X) \longrightarrow \mathbb{CP}^1$ we can lift the standard complex structure I_0 on $\mathbb{CP}^1$ to $\mathcal{T}(X)$ and denote it by $p^* I_0$, and define the complex structure on $\mathcal{T}(X)$ by $J = I_z + p^* I_0$. Of course, this makes the map $p : \mathcal{T}(X) \longrightarrow \mathbb{CP}^1$ holomorphic. Furthermore, we have a double fibration, a la Penrose[4], (cf. Diagram 12.6.4 of [BG08])

$$\begin{array}{ccccc} & & \mathcal{T}(X) & & \\ & \overset{p}{\swarrow} & & \searrow & \\ \mathbb{CP}^1 & & \rightsquigarrow & & X \end{array} \tag{26}$$

which gives a correspondence: points $z \in \mathbb{CP}^1 \simeq S^2$ correspond to complex structures I_z on X in the given hyperkähler structure $\mathfrak{I}$; points

[4]This arises from Penrose's nonlinear graviton [Pen76] and is amply treated in books [Wel82, WW90, MW96].

$x \in X$ correspond to rational curves in $\mathcal{T}(X)$ with normal bundle $2n\mathcal{O}(1)$, called *twistor lines*. The general point is that the holomorphic data on the twistor space $\mathcal{T}(X)$ encodes the hyperkähler data on X. We note that generally the twistor space $(\mathcal{T}(X), J)$ is not Kähler.

Let (ξ, η, Φ_0, g) be a Sasakian structure which is associated to the transverse hyperkähler structure $\{I_i\}_{i=1}^3$. Then by Lemma 4.7 ω_1^T is a transverse Kähler form satisfying $\omega_1^T = d\eta$. Moreover, $\omega_+ = \omega_2 + i\omega_3$ is a transverse holomorphic section of $H^{2,0}(\mathcal{F}_\xi)$ and $\omega_- = \omega_2 - i\omega_3$ is transverse anti-holomorphic section of $H^{0,2}(\mathcal{F}_\xi)$. So $h^{2,0} \neq 0 \neq h^{0,2}$. Since these transverse hyperkähler structures are null Sasakian, the transverse geometry is that of hyperkähler orbifolds X with cyclic isotropy groups. Now let us deform the complex structure of X in a disc in S^2 centered around I_1. This gives the twisted $(2,0)$ form

$$\omega_z = \omega_+ + 2z\omega_1 - z^2\omega_- \tag{27}$$

as a section of $p^*\mathcal{O}(2) \otimes \Omega^2(\mathcal{T}(X))$ and representing the variation of Hodge structures on X. Thus, ω_z has two interpretations: (1) as a holomorphic $(2,0)$-form on the twistor space, and (2) as a holomorphic $(2,0)$-form on each member X_z of the family of complex manifolds (orbifolds) parameterized by a holomorphic section of $\mathcal{O}(2)$ on $\mathbb{CP}^1$. Now ω_z defines a class in $H^2_{orb}(X_z, \mathbb{Q})$ for at most a countable number of $z \in \mathbb{CP}^1$. Thus, for only a countable number of points $z \in \mathbb{CP}^1$ will $[\omega_z]$ lie in an integral Hodge lattice $\Lambda = H^2_{orb}(X_z, \mathbb{Z})$. Transversally, such integer lattices in $H^{1,1}_B(\mathcal{F}_\xi)$ are constructed by Proposition 3.11 from the transverse Picard group $\mathrm{Pic}^T(M, \mathcal{F}_\xi)$ of isomorphism classes of orbi-line-bundles over X_z at least when $b_1(M) = 0$ which holds in our case. Then since for compact irreducible hyperkähler orbifolds $h^{2,0} = h^{0,2} = 1$ [Fuj83], it follows that in the case when $[\omega_z] \notin H^2_{orb}(X_z, \mathbb{Z})$ then the transverse complex structure admits no integer lattice in $H^{1,1}_B(\mathcal{F}_\xi)$. Thus, in such cases $\mathrm{Pic}^T(M, \mathcal{F}_\xi) = \mathrm{Pic}^{orb}(X_z) = 0$. So for all but a countable number of points $z \in \mathbb{CP}^1$ we have $\mathrm{Pic}^T(M, \mathcal{F}_\xi)) = 0$ and so all but a countable number of points $z \in \mathbb{CP}^1$ are non-algebraic hyperkähler orbifolds. These cannot represent the transverse Kähler structure of a Sasakian structure, since null Sasakian structures are algebraic, that is they are the total space of an orbibundle over a projective algebraic variety [BG08]. □

4.3. **Trivial Restricted Transverse Holonomy.** Finally, we briefly consider the case when the restricted transverse holonomy group $\mathrm{Hol}^0(\mathcal{F})$ is the identity. For simplicity we only consider the regular case of S^1 bundles over a polarized Abelian variety. In this case the restricted transverse holonomy group is the identity. They are nilmanifolds $\mathfrak{N}_1$

of dimension $2n+1$ formed as quotients of the $(2n+1)$-dimensional Heisenberg group $\mathfrak{H}(\mathbb{R})$ by a lattice subgroup $\Gamma_{\mathbf{l}}$ where $\mathbf{l} = (l_1, \ldots, l_n)$ is a $\mathbb{Z}$-vector whose components are positive and satisfy the divisibility conditions $l_j | l_{j+1}$ for $j = 1, \ldots, n-1$ [Fol04]. Now $\mathfrak{N}_{\mathbf{l}}$ has a canonical strictly pseudoconvex CR structure $(\mathcal{D}, J)$. In fact it has a compatible Sasakian structure $\mathcal{S}_{\mathbf{l}}$, unique up to equivalence, and $\mathcal{S}_{\mathbf{l}}$ has constant Φ-sectional curvature -3 [Boy09]. These nilmanifolds are both homogeneous and Sasakian, but they are not Sasaki homogeneous. Moreover, Folland shows that there is a 1-1 correspondence between equivalence classes of such CR structures and polarized Abelian varieties $(\mathbb{C}^n/\Lambda_{\mathbf{l}}, L)$ equipped with a positive line bundle L where the lattice $\Lambda_{\mathbf{l}}$ is the image of $\Gamma_{\mathbf{l}}$ under the natural projection $\pi : \mathfrak{N}_{\mathbf{l}} \longrightarrow \mathbb{C}^n/\Lambda_{\mathbf{l}}$. Here l_1 is the largest positive integer such that $c_1(L)/l_1$ is primitive in $H^2(\mathbb{C}^n/\Lambda_{\mathbf{l}}, \mathbb{Z})$. The first homology group of such nilmanifolds is $H_1(\mathfrak{N}_{\mathbf{l}}, \mathbb{Z}) = \mathbb{Z}^{2n} + \mathbb{Z}_{l_1}$. Nozawa [Noz14] proved that all such $(\mathfrak{N}_{\mathbf{l}}, \mathcal{S}_{\mathbf{l}})$ are $\mathcal{S}$-unstable when $n \geq 2$.

4.4. **Reducible Transverse Kähler Holonomy.** Here we consider the case of reducible Kähler holonomy, namely, the *join* of two quasiregular Sasaki manifolds M_1, M_2 defined in [BG00, BGO07] and developed further in [BHLTF18]. Recall that for any pair of relatively prime positive integers $(l_1, l_2) = \mathbf{l}$ we define the join $M \star_{\mathbf{l}} M_2$ of two quasiregular Sasaki manifolds M_1, M_2 with Reeb vector fields ξ_1, ξ_2 respectively, by the quotient of $M_1 \times M_2$ by the S^1 generated by the vector

$$L_{\mathbf{l}} = \frac{1}{2l_1}\xi_1 - \frac{1}{2l_2}\xi_2 \tag{28}$$

where ξ_i is the Reeb field of the Sasakian structure on M_i. This gives rise to the commutative diagram

$$\begin{array}{ccc} M_1 \times M_2 & & \\ & \searrow \pi_L & \\ \Big\downarrow \pi_2 & & M_1 \star_{\mathbf{l}} M_2 \\ & \swarrow \pi_1 & \\ N_1 \times N_2 & & \end{array} \tag{29}$$

where N_i are the quotient orbifolds of M_i, and the Reeb vector field of the induced Sasakian structure on $M_1 \star_{\mathbf{l}} M_2$ is given by

$$\xi_{\mathbf{l}} = \frac{1}{2l_1}\xi_1 + \frac{1}{2l_2}\xi_2. \tag{30}$$

We now have

Corollary 4.10. *Let $\mathcal{M}_{\mathfrak{l}} = M_1 \star_{\mathfrak{l}} M_2$ be the join of quasiregular Sasaki manifolds M_i, $i = 1, 2$. Suppose also that $b_1(M_i) = 0$ for $i = 1, 2$, and the basic Hodge numbers $h^{0,2}_B(M_i)$ also vanish. Then every Sasakian structure in the Sasaki cone $\mathfrak{t}^+_{\mathfrak{l}}$ of $\mathcal{M}_{\mathfrak{l}}$ is $\mathcal{S}$-stable.*

Proof. Since by [GNT16] the basic Hodge numbers depend only on the underlying CR structure, it suffices to prove the corollary for the Reeb field (30) which is quasiregular. Using Proposition 2.19 we can compute the Hodge numbers of the product orbifold $N_1 \times N_2$. By the Hodge-Kunneth formula ([Voi02] page 286) we have

$$H^{0,2}(N_1 \times N_2) = H^{0,2}(N_1) \otimes H^{0,0}(N_2) + H^{0,0}(N_1) \otimes H^{0,2}(N_2) + H^{0,1}(N_1) \otimes H^{0,1}(N_2).$$

This implies that

$$h^{0,2}_B(\mathcal{M}_{\mathfrak{l}}) = h^{0,2}_B(M_1) + h^{0,2}_B(M_2) + h^{0,1}_B(M_1) h^{0,1}_B(M_2)$$

which vanishes by hypothesis and the injectivity of $H^1_B \longrightarrow H^1(M)$. The result then follows from Theorem 1.2. □

We remark that the hypothesis of the corollary implies, using Theorem 1.2 that the Sasakian structures on M_i are both $\mathcal{S}$-stable. However, we do not know whether generally the join of $\mathcal{S}$-stable Sasakian structures is $\mathcal{S}$-stable.

4.5. **Fiber Joins and $\mathcal{S}$-Stability.** There is another type of join construction due to Yamazaki [Yam99] which describes a construction of K-contact structures on sphere bundles over a symplectic manifold. Given a compact symplectic manifold N with $d+1$ integral symplectic forms ω_j, not necessarily distinct. Let L_j be the complex line bundle on N such that $c_1(L_j) = [\omega_j]$, then Yamazaki shows that the unit sphere bundle in the complex vector bundle $\oplus_{j=1}^{d+1} L_j^*$ has a natural K-contact structure associated to each Reeb vector field in the Sasaki cone $\mathfrak{t}^+_{sph}$ of the sphere S^{2d+1}. The manifold is denoted by $M = M_1 \star_f \cdots \star_f M_{d+1}$ where M_j is principal S^1 bundle associated to L_j. Moreover, it is easy to see that this K-contact structure is Sasakian if N is a projective variety and ω_j are integral Kähler forms [BTF20]. It was also shown there that such Sasakian structures come in two types, cone decomposable fiber joins and cone indecomposable fiber joins. The former is equivalent to a special case of the joins described in Section 4.4; however, it follows from Proposition 3.8 (2) of [BTF20] that the cone indecomposable fiber joins have irreducible $U(n)$ transverse holonomy. Nevertheless, in either case we have

Corollary 4.11. *Let N be a smooth projective algebraic variety with $b_1(N) = 0$ and integral Kähler forms ω_j and let $M = M_1 \star_f \cdots \star_f M_{d+1}$ be* *a fiber join with its spherical Sasaki subcone $\mathfrak{t}^+_{sph}$ of Sasakian structures on M. Assume also that the Hodge number $h^{0,2}(N)$ vanishes. Then every Sasakian structure in $\mathfrak{t}^+_{sph}$ is $\mathcal{S}$-stable.*

Proof. Again since the basic Hodge numbers depend only on the underlying CR structure [GNT16], we can choose the regular Reeb vector field in $\mathfrak{t}^+_{sph}$. Then from the Leray-Hirsch theorem there is an isomorphism of groups (not necessarily of rings)

$$H^*(\mathbb{P}(\oplus_{j=1}^{d+1}L_j^*)) \approx H^*(M)\otimes H^*(\mathbb{CP}^d).$$

Applying the Hodge decomposition to each piece gives

$$H^{0,2}(\mathbb{P}(\oplus_{j=1}^{d+1}L_j^*)) = H^{0,2}(N)\otimes H^{0,0}(\mathbb{CP}^d)\oplus H^{0,1}(N)\otimes H^{0,1}(\mathbb{CP}^d)\oplus H^{0,0}(N)\otimes H^{0,2}(\mathbb{CP}^d).$$

But this clearly implies $h^{0,2}(\mathbb{P}(\oplus_{j=1}^{d+1}L_j^*)) = h^{0,2}(N)$, so the corollary follows from Theorem 1.2. □

References

[AC18] Vestislav Apostolov and David M.J. Calderbank, *The CR geometry of weighted extremal Kähler and Sasaki metrics*, arXiv:1810.10618 (2018).

[ALR07] Alejandro Adem, Johann Leida, and Yongbin Ruan, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics, vol. 171, Cambridge University Press, Cambridge, 2007. MR 2359514

[Bea83] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782 (1984). MR 730926 (86c:32030)

[BG99] C. P. Boyer and K. Galicki, *3-Sasakian manifolds*, Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom., VI, Int. Press, Boston, MA, 1999, pp. 123–184. MR 2001m:53076

[BG00] Charles P. Boyer and Krzysztof Galicki, *On Sasakian-Einstein geometry*, Internat. J. Math. **11** (2000), no. 7, 873–909. MR 1792957

[BG08] ______, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008. MR MR2382957 (2009c:53058)

[BGK05] C. P. Boyer, K. Galicki, and J. Kollár, *Einstein metrics on spheres*, Ann. of Math. (2) **162** (2005), no. 1, 557–580. MR 2178969 (2006j:53058)

[BGM06] C. P. Boyer, K. Galicki, and P. Matzeu, *On eta-Einstein Sasakian geometry*, Comm. Math. Phys. **262** (2006), no. 1, 177–208. MR 2200887 (2007b:53090)

[BGO07] Charles P. Boyer, Krzysztof Galicki, and Liviu Ornea, *Constructions in Sasakian geometry*, Math. Z. **257** (2007), no. 4, 907–924. MR MR2342558 (2008m:53103)

[BGS08] Charles P. Boyer, Krzysztof Galicki, and Santiago R. Simanca, *Canonical Sasakian metrics*, Commun. Math. Phys. **279** (2008), no. 3, 705–733. MR MR2386725

[BHLTF18] Charles P. Boyer, Hongnian Huang, Eveline Legendre, and Christina W. Tønnesen-Friedman, *Reducibility in Sasakian geometry*, Trans. Amer. Math. Soc. **370** (2018), no. 10, 6825–6869. MR 3841834

[Bog78] F. A. Bogomolov, *Hamiltonian Kählerian manifolds*, Dokl. Akad. Nauk SSSR **243** (1978), no. 5, 1101–1104. MR 514769

[Boy09] Charles P. Boyer, *The Sasakian geometry of the Heisenberg group*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **52(100)** (2009), no. 3, 251–262. MR MR2554644

[BTF20] Charles P. Boyer and Christina W. Tønnesen-Friedman, *Sasakian geometry on sphere bundles*, arXiv:2007.10236 (2020).

[Cam04] F. Campana, *Orbifolds, special varieties and classification theory*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 3, 499–630. MR 2097416

[Car84] Y. Carrière, *Flots riemanniens*, Astérisque (1984), no. 116, 31–52, Transversal structure of foliations (Toulouse, 1982). MR 86m:58125a

[CMST20] Alejandro Cañas, Vicente Muñoz, Mattias Schütt, and Aleksy Tralle, *Quasi-regular Sasakian and K-contact structures on Smale-Barden manifolds*, arXiv:2004.12643 (2020).

[CV14] Jaime Cuadros Valle, *Null Sasaki η-Einstein structures in 5-manifolds*, Geom. Dedicata **169** (2014), 343–359. MR 3175253

[DK79] T. Duchamp and M. Kalka, *Deformation theory for holomorphic foliations*, J. Differential Geom. **14** (1979), no. 3, 317–337 (1980). MR 82b:57019

[DK80] ______, *Stability theorems for holomorphic foliations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 255–266. MR 81f:57022

[EKA88] Aziz El Kacimi-Alaoui, *Stabilité des V-variétés kahlériennes*, Holomorphic dynamics (Mexico, 1986), Lecture Notes in Math., vol. 1345, Springer, Berlin, 1988, pp. 111–123. MR 980955

[EKA90] A. El Kacimi-Alaoui, *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math. **73** (1990), no. 1, 57–106. MR 91f:58089

[EKAG97] A. El Kacimi Alaoui and B. Gmira, *Stabilité du caractère kählérien transverse*, Israel J. Math. **101** (1997), 323–347. MR 1484881

[EKAH86] A. El Kacimi-Alaoui and G. Hector, *Décomposition de Hodge basique pour un feuilletage riemannien*, Ann. Inst. Fourier (Grenoble) **36** (1986), no. 3, 207–227. MR 87m:57029

[Fol04] G. B. Folland, *Compact Heisenberg manifolds as CR manifolds*, J. Geom. Anal. **14** (2004), no. 3, 521–532. MR MR2077163 (2005d:32057)

[Fuj83] Akira Fujiki, *On primitively symplectic compact Kähler V-manifolds of dimension four*, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 71–250. MR 728609

[GHJ03] M. Gross, D. Huybrechts, and D. Joyce, *Calabi-Yau manifolds and related geometries*, Universitext, Springer-Verlag, Berlin, 2003, Lectures from the Summer School held in Nordfjordeid, June 2001. MR 1963559 (2004c:14075)

[GHS83] J. Girbau, A. Haefliger, and D. Sundararaman, *On deformations of transversely holomorphic foliations*, J. Reine Angew. Math. **345** (1983), 122–147. MR 84j:32026

[GM80] X. Gómez-Mont, *Transversal holomorphic structures*, J. Differential Geom. **15** (1980), no. 2, 161–185 (1981). MR 82j:53065

[GNT16] Oliver Goertsches, Hiraku Nozawa, and Dirk Töben, *Rigidity and vanishing of basic Dolbeault cohomology of Sasakian manifolds*, J. Symplectic Geom. **14** (2016), no. 1, 31–70. MR 3523249

[Got04] Ryushi Goto, *Moduli spaces of topological calibrations, Calabi-Yau, hyper-Kähler,* G_2 *and* Spin(7) *structures*, Internat. J. Math. **15** (2004), no. 3, 211–257. MR 2060789

[Hit87] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126. MR 887284 (89a:32021)

[HKLR87] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), no. 4, 535–589. MR 88g:53048

[Huy99] Daniel Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. **135** (1999), no. 1, 63–113. MR 1664696

[HV15] Georges Habib and Luigi Vezzoni, *Some remarks on Calabi-Yau and hyper-Kähler foliations*, Differential Geom. Appl. **41** (2015), 12–32. MR 3353736

[Joy07] Dominic D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics, vol. 12, Oxford University Press, Oxford, 2007. MR 2292510

[Kod60] K. Kodaira, *On deformations of some complex psuedo-group structures*, Ann. of Math. (2) **71** (1960), 224–302. MR 22 #5992

[Kol05] J. Kollár, *Einstein metrics on five-dimensional Seifert bundles*, J. Geom. Anal. **15** (2005), no. 3, 445–476. MR 2190241

[KS60] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures. III. Stability theorems for complex structures*, Ann. of Math. (2) **71** (1960), 43–76. MR 22 #5991

[KS61] ———, *Multifoliate structures*, Ann. of Math. (2) **74** (1961), 52–100. MR 0148086 (26 #5595)

[Kur71] M. Kuranishi, *Deformations of compact complex manifolds*, Les Presses de l'Université de Montréal, Montreal, Que., 1971, Séminaire de Mathématiques Supérieures, No. 39 (Été 1969). MR 50 #7588

[MM03] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003. MR 2012261

[Moe02] I. Moerdijk, *Orbifolds as groupoids: an introduction*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 205–222. MR 2004c:22003

[Mol79] P. Molino, *Feuilletages riemanniens sur les variétés compactes; champs de Killing transverses*, C. R. Acad. Sci. Paris Sér. A-B **289** (1979), no. 7, A421–A423. MR 80j:57026

[Mol82] ———, *Géométrie globale des feuilletages riemanniens*, Nederl. Akad. Wetensch. Indag. Math. **44** (1982), no. 1, 45–76. MR 84j:53043

[Mol88] ———, *Riemannian foliations*, Progress in Mathematics, vol. 73, Birkhäuser Boston Inc., Boston, MA, 1988, Translated from the French by Grant Cairns, With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu. MR 89b:53054

[Mor10] Takayuki Moriyama, *Deformations of transverse Calabi-Yau structures on foliated manifolds*, Publ. Res. Inst. Math. Sci. **46** (2010), no. 2, 335–357. MR 2722781

[MP97] I. Moerdijk and D. A. Pronk, *Orbifolds, sheaves and groupoids*, K-Theory **12** (1997), no. 1, 3–21. MR 98i:22004

[MS85] P. Molino and V. Sergiescu, *Deux remarques sur les flots riemanniens*, Manuscripta Math. **51** (1985), no. 1-3, 145–161. MR 86h:53035

[Muk88] Shigeru Mukai, *Finite groups of automorphisms of $K3$ surfaces and the Mathieu group*, Invent. Math. **94** (1988), no. 1, 183–221. MR 958597

[MW96] L. J. Mason and N. M. J. Woodhouse, *Integrability, self-duality, and twistor theory*, London Mathematical Society Monographs. New Series, vol. 15, The Clarendon Press, Oxford University Press, New York, 1996, Oxford Science Publications. MR 1441309

[Nik76] V. V. Nikulin, *Finite groups of automorphisms of Kählerian surfaces of type $K3$*, Uspehi Mat. Nauk **31** (1976), no. 2(188), 223–224. MR 0409904

[Noz14] Hiraku Nozawa, *Deformation of Sasakian metrics*, Trans. Amer. Math. Soc. **366** (2014), no. 5, 2737–2771. MR 3165654

[Pen76] R. Penrose, *Nonlinear gravitons and curved twistor theory*, General Relativity and Gravitation **7** (1976), no. 1, 31–52, The riddle of gravitation–on the occasion of the 60th birthday of Peter G. Bergmann (Proc. Conf., Syracuse Univ., Syracuse, N. Y., 1975). MR 0439004 (55 #11905)

[Sar85] M. Saralegui, *The Euler class for flows of isometries*, Differential geometry (Santiago de Compostela, 1984), Res. Notes in Math., vol. 131, Pitman, Boston, MA, 1985, pp. 220–227. MR 864872

[Tia87] Gang Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, pp. 629–646. MR 915841

[TV08] Adriano Tomassini and Luigi Vezzoni, *Contact Calabi-Yau manifolds and special Legendrian submanifolds*, Osaka J. Math. **45** (2008), no. 1, 127–147. MR 2416653

[Ver05] M. Verbitsky, *Hypercomplex structures on Kähler manifolds*, Geom. Funct. Anal. **15** (2005), no. 6, 1275–1283. MR 2221248

[Voi02] C. Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2002, Translated from the French original by Leila Schneps. MR 1967689 (2004d:32020)

[Wel82] R. O. Wells, Jr., *Complex geometry in mathematical physics*, Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], vol. 78, Presses de l'Université de Montréal, Montreal, Que., 1982, Notes by Robert Pool. MR 654864

[WW90] R. S. Ward and Raymond O. Wells, Jr., *Twistor geometry and field theory*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1990. MR 1054377

[Yam99] T. Yamazaki, *A construction of K-contact manifolds by a fiber join*, Tohoku Math. J. (2) **51** (1999), no. 4, 433–446. MR 2001e:53094

Charles P. Boyer, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131.
E-mail addresses: cboyer@unm.edu

Hongnian Huang, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131.
E-mail addresses: hnhuang@unm.edu

Christina W. Tønnesen-Friedman, Department of Mathematics, Union College, Schenectady, New York 12308, USA
E-mail addresses: tonnesec@union.edu