Elements of Calculus I, MATH 180 Midterm 3

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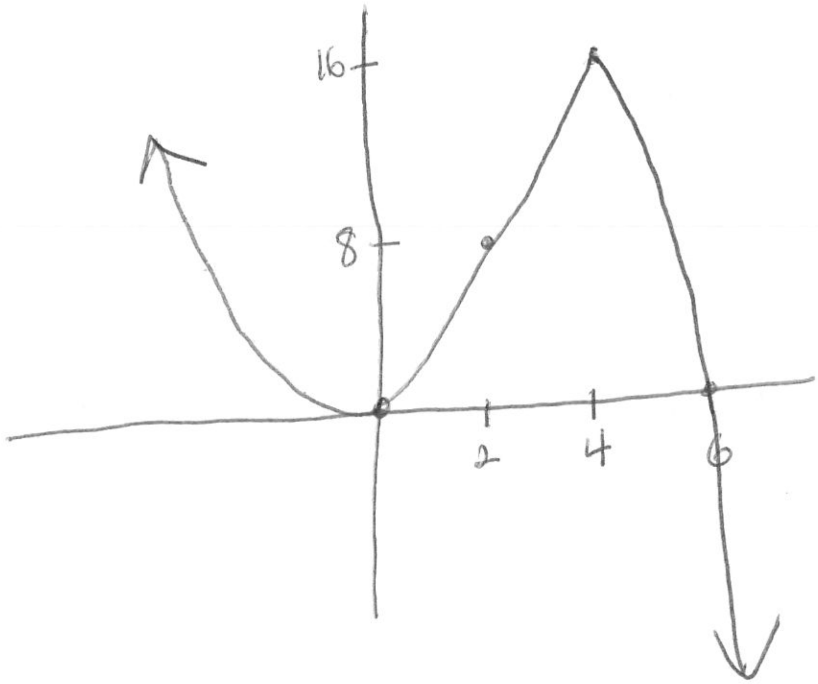
(1) (10 points) Suppose $f(x)$ is a function satisfying the following properties:
- $f'(x) > 0$ on $(0, 4)$,
- $f'(x) < 0$ on $(-\infty, 0)$ and $(4, \infty)$,
- $f''(x) > 0$ on $(-\infty, 4)$,
- $f''(x) < 0$ on $(4, \infty)$,
- $(0, 0)$, $(2, 8)$, $(4, 16)$ and $(6, 0)$ are points on the graph.

Graph $f(x)$.



(2) (20 points) Given $f(x) = 5 + 4x - x^2$.

(a) Find both $f'(x)$ and $f''(x)$.

$f'(x) = 4 - 2x \quad f''(x) = -2$

(b) Determine the intervals where $f(x)$ is increasing/decreasing.

$+ + + + \; 2 \; - - - -$

increasing $(-\infty, 2)$ decreasing $(2, \infty)$

(c) Determine any points where $f(x)$ has a relative maximum or minimum.

There is a relative max at $(2, f(2))$

$f(2) = 5 + 8 - 4 = 9$ so $(2, 9)$

(d) Determine the intervals where $f(x)$ is concave up/concave down.

Since $f''(x)$ is negative

$f(x)$ is concave down for all reals.

(e) Determine any points where $f(x)$ has an inflection point.

$f(x)$ has no inflection point.

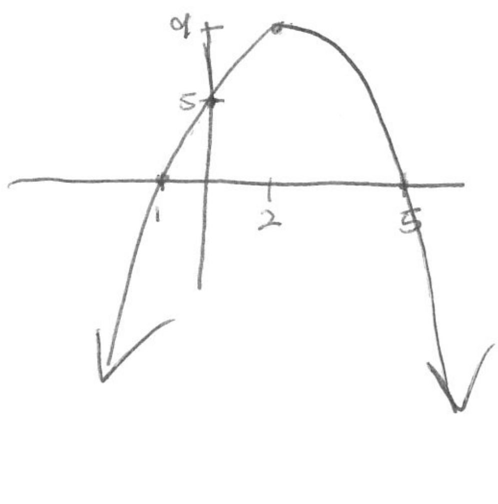
(f) Determine the x and y intercepts.

$0 = 5 + 4x - x^2 = (5 - x)(1 + x)$

so $(+5, 0)$ & $(-1, 0)$ x intercepts

$(0, 5)$ a y-intercept.

(g) Graph $f(x)$.



(3) (10 points) Find the points where $f(x) = x^3 - 12x + 3$ has an absolute maximum or an absolute minimum on $[0, 3]$.

$$f'(x) = 3x^2 - 12 = 3(x+2)(x-2)$$

$x = -2$ $\boxed{x = 2}$ only one in $[0,3]$

$f(0) = 3$

$f(2) = 8 - 24 + 3 = -19$

$f(3) = 27 - 36 + 3 = -6$

$(0,3)$ is where the max is

$(2,-19)$ is where the min is.

(4) (10 points) Show that the function $f(x) = \dfrac{x}{x^2+4}$ has an absolute maximum and an absolute minimum on the real line. (Hint: You will have to look at the behavior of the function when $|x|$ is large.)

$$f'(x) = \frac{x^2 + 4 - x(2x)}{(x^2+4)^2} = \frac{4 - x^2}{(x^2+4)^2} = 0$$

when $x = \pm 2$

$$f(2) = \frac{2}{4+4} = \frac{1}{4}$$

$$f(-2) = \frac{-2}{4+4} = -\frac{1}{4}$$

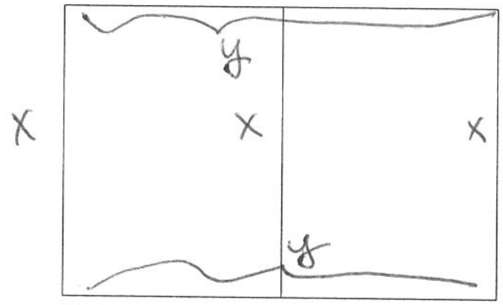
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{x^2+4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{\overset{\to 0}{\frac{1}{x}}}{1 + \underset{\to 0}{4/x^2}} = 0$$

$\lim_{x \to \infty} f(x) = 0$ also

so the max is at $(2, \frac{1}{4})$

& the min is at $(-2, -\frac{1}{4})$.

(5) (10 points) Two adjoining pony enclosures are to be made in a field. The following diagram illustrates the placing of the fencing material.



If there are 48 yards of fencing material, what are the dimensions of the enclosure which provides the most room for the ponies?

$3x + 2y = 48$ so $y = \frac{48}{2} - \frac{3}{2}x$

$\downarrow$

$0 \leq x \leq \frac{48}{3} = 16$ $\qquad = 24 - \frac{3}{2}x$

$A = xy = x(24 - \frac{3}{2}x) = 24x - \frac{3}{2}x^2$

$A'(x) = 24 - 3x = 0$

$\Rightarrow x = 8$

$A(0) = 0$

$A(8) = 24 \cdot 8 - \frac{3}{2} \cdot 64 = 192 - 96$

$= 96$

$A(16) = 0$

so max when $x = 8$

so dimensions of total enclosure are 8 by 12

or the dimensions of each smaller enclosure is 8 by 6.

(6) Find the derivatives of the following: (5 points each)

(a) $f(x) = e^{x^2}$.

$$f'(x) = e^{x^2}(2x)$$

(b) $f(x) = x\ln(2x+1)$.

$$f'(x) = x \cdot \frac{2}{2x+1} + \ln(2x+1)$$

$$= \frac{2x}{2x+1} + \ln(2x+1)$$

(7) Determine the following: (5 points each)

(a) $\int(x^5 - 2x^3 + 4x + 1)dx$

$$= \frac{1}{6}x^6 - \frac{1}{2}x^4 + 2x^2 + x + C$$

(b) $\int(e^{7x} + \frac{1}{7x})dx$ $= \frac{1}{7}\int e^{7x} + \frac{1}{7x}\, 7dx$

$u = 7x$

$du = 7dx$

$$= \frac{1}{7}\int e^{u} + \frac{1}{u}\, du$$

$$= \frac{1}{7}(e^{u} + \ln|u|) + C$$

$$= \frac{1}{7}(e^{7x} + \ln|7x|) + C.$$

(8) (5 points) Verify that $F(x) = \ln(x^3+1) - 5x^4 + 20e^{5x}$ is an antiderivative of

$$f(x) = \frac{3x^2}{x^3+1} - 20x^3 + 100e^{5x}.$$

$$F'(x) = \frac{3x^2}{x^3+1} - 5\cdot 4x^3 + 20\cdot 5e^{5x}$$
$$= \frac{3x^2}{x^3+1} - 20x^3 + 100e^{5x}$$
$$= f(x).$$

(9) (10 points) If $f'(x) = 3\sqrt{x} - 2x$ and $f(1) = 3$, find $f(x)$.

$$f(x) = \int 3\sqrt{x} - 2x\,dx$$
$$= \int 3x^{1/2} - 2x\,dx$$
$$= 3\cdot\frac{2}{3}x^{3/2} - x^2 + C$$
$$3 = f(1) = 2\cdot 1^{3/2} - 1^2 + C$$
$$3 = 1 + C \quad \text{so } C = 2$$
$$f(x) = 2x^{3/2} - x^2 + 2$$

(10) (5 points) If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, how are $F(x)$ and $G(x)$ related?

$F(x)$ & $G(x)$ differ by a constant, ie.

$F(x) = G(x) + C$.