Solutions

MATH 314–Linear Algebra
Midterm 1
Dr. Janet Vassilev

(1) Solve the following system of equations: out of 15 pts

$$
\begin{aligned}
2x_1 + 3x_2 - 4x_3 + x_4 &= 6\\
x_1 - 2x_2 + 5x_3 + 3x_4 &= 2\\
4x_1 - x_2 + 6x_3 + 5x_4 &= 8
\end{aligned}
$$

$$
\left(\begin{array}{cccc|c} 2 & 3 & -4 & 1 & 6\\ 1 & -2 & 5 & 3 & 2\\ 4 & -1 & 6 & 5 & 8\end{array}\right)
\xrightarrow{R_1 \leftrightarrow R_2}
\left(\begin{array}{cccc|c} 1 & -2 & 5 & 3 & 2\\ 2 & 3 & -4 & 1 & 6\\ 4 & -1 & 6 & 5 & 8\end{array}\right)
$$

$$
\xrightarrow[-4R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2}
\left(\begin{array}{cccc|c} 1 & -2 & 5 & 3 & 2\\ 0 & 7 & -14 & -5 & 2\\ 0 & 7 & -14 & -7 & 0\end{array}\right)
$$

$$
\xrightarrow{-R_2+R_3 \to R_3}
\left(\begin{array}{cccc|c} 1 & -2 & 5 & 3 & 2\\ 0 & 7 & -14 & -5 & 2\\ 0 & 0 & 0 & -2 & -2\end{array}\right)
$$

$$
\xrightarrow{-\frac{1}{2}R_3}
\left(\begin{array}{cccc|c} 1 & -2 & 5 & 3 & 2\\ 0 & 7 & -14 & -5 & 2\\ 0 & 0 & 0 & 1 & 1\end{array}\right)
$$

$$
\xrightarrow[-3R_3+R_1 \to R_1]{5R_3+R_2 \to R_2}
\left(\begin{array}{cccc|c} 1 & -2 & 5 & 0 & -1\\ 0 & 7 & -14 & 0 & 7\\ 0 & 0 & 0 & 1 & 1\end{array}\right)
$$

$$
\xrightarrow{\frac{1}{7}R_2}
\left(\begin{array}{cccc|c} 1 & -2 & 5 & 0 & -1\\ 0 & 1 & -2 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\end{array}\right)
$$

$$
\xrightarrow{2R_2+R_1 \to R_1}
\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1\\ 0 & 1 & -2 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\end{array}\right)
$$

Let $x_3 = \alpha$ & $x_4 = 1$ (by R_3) $x_2 - 2\alpha = 1 \Rightarrow x_2 = 1 + 2\alpha$
and $x_1 + \alpha = 1$ so $x_1 = 1 - \alpha$ $\boxed{(1-\alpha, 1+2\alpha, \alpha, 1)}$

(2) If possible, find solutions for the following systems of equations represented by augmented matrices. Label each system consistent or inconsistent.

(out of 15pts)

(a) $A = \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$

Let $x_3 = \alpha$ $x_4 = 1$

$x_2 = -2\alpha$ & $x_1 = -\alpha$

so $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\alpha \\ -2\alpha \\ \alpha \\ 1 \end{pmatrix}$

& system is consistent.

(b) $B = \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right)$

Since $0 \neq 1$ the last row implies the system is inconsistent.

(c) $C = \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 8 \end{array}\right)$

If $x_1 = 3$ and $x_2 = 2$

then $2 \cdot 3 + 1 \cdot 2 = 8$

satisfies row 3

so the system has solution $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

and is consistent.

(out of 12 pts)

(3) If $A = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 0 \end{pmatrix}$, determine:

(a) AB

$$\begin{pmatrix} 1 & 3 & 4 \\ 5 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 11 & 15 \\ -2 & 15 \end{pmatrix}$$

(b) $2B$

$$2 \begin{pmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 2 & 8 \\ 4 & 0 \end{pmatrix}$$

(c) BA

$$\begin{pmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 4 \\ 5 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 15 & 0 & -3 \\ 21 & 3 & 0 \\ 2 & 6 & 8 \end{pmatrix}$$

(d) $A + B^T$

$$\begin{pmatrix} 1 & 3 & 4 \\ 5 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 6 \\ 8 & 4 & -1 \end{pmatrix}$$

(4) Using the properties of matrices, show that for any $n \times n$ matrices A and B that $AB^T + BA^T$ is symmetric.

$C = AB^T + BA^T$ If $C = C^T$ C is symmetric.

$(AB^T + BA^T)^T \underset{\text{sum of transposes}}{=} (AB^T)^T + (BA^T)^T \overset{\text{product of transposes}}{=} (B^T)^T A^T + (A^T)^T B^T$

$\underset{\text{double transposes}}{=} BA^T + AB^T = AB^T + BA^T$

so $AB^T + BA^T$ symmetric.

(5) Find the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -6 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & -4 & -6 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{4R_2+R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & -2 & 1 & 4 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & 2 \end{array}\right)$$

$$\xrightarrow[-3R_3+R_1 \to R_1]{-2R_3+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -\frac{3}{2} & -6 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & 2 \end{array}\right)$$

$$\xrightarrow{-2R_2+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & 2 \end{array}\right)$$

so $A^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 2 & -1 & -3 \\ -1 & \frac{1}{2} & 2 \end{pmatrix}$

you can also compute $\frac{1}{\det A} \operatorname{adj} A$ to get the same.

(6) Let A be a 4×4 matrix and $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. If $A\mathbf{x} = \mathbf{0}$, what are two things that you can say about A? (There are more than two possibilities but I just want two.) (6 pts)

Since $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a nontrivial solution to $A\bar{x} = \bar{0}$ then A is singular and the determinant of A is 0. Also A is not row equivalent to the identity and $A\bar{y} = \bar{b}$ either is inconsistent or doesn't have a unique solution.

(7) If X is a 2×2 matrix and $A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}$, solve for X if $AX = B$.

(15 pts)

$$A^{-1} = \frac{1}{4-3}\begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$$

so $X = A^{-1}B = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}$

$$= \begin{pmatrix} -7 & -9 \\ 3 & 3 \end{pmatrix}$$

(8) Find the determinant of $\begin{pmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 & 4 \end{pmatrix}$ (10 pts)

$$A = \begin{pmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{-R_1+R_5 \to R_5} \begin{pmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & -2 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow{-1/2 R_2 + R_4 \to R_4} \begin{pmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow{2/3 R_3 + R_5 \to R_5} \begin{pmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} = B$$

$\det B = 1 \cdot 2 \cdot 3 \cdot 2 \cdot 3 = 36$

We obtained B from A by type III row ops so $\det A = 36$ too.

(9) Let A and B be 2×2 matrices with $|A| = 4$ and $|B| = \frac{3}{2}$, determine $|AB|$ and $|3B^{-1}|$.

(6pts)

$$|AB| = |A||B| = 4 \cdot 3/2 = 6$$

$$|3B^{-1}| = 3^2 \cdot \frac{1}{|B|} = 9 \cdot \frac{2}{3} = 6$$