Solutions

MATH 314–Linear Algebra
Midterm 2
Dr. Janet Vassilev

(1) Is $\mathbb{R}^2$ with the usual scalar multiplication $\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$ and "addition" defined as $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ y_2 \end{pmatrix}$ a vector space? Justify your answer.

5 pts $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ y_2 \end{pmatrix} \neq \begin{pmatrix} y_1 + x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

fails the commutative property

so $\mathbb{R}^2$ with the above scalar mult & addition is not a vector space

(2) Determine if the following subsets are subspaces. In each case, justify your answer.

(a) $S = \{(x, x-y, x+y)^T \mid x, y \text{ real}\}$ in $\mathbb{R}^3$?

$(x, x-y, x+y) + (z, z-w, z+w) = (x+z, x+z-y-w, x+z+y+w) \in S$ (so + is closed)

5pts $\alpha(x, x-y, x+y) = (\alpha x, \alpha x - \alpha y, \alpha x + \alpha y) \in S$ (so scalar mult closed)

Thus S is a subspace.

(b) $T = \{ax + 1 \mid a \text{ real}\}$ in P_1?

$\alpha(ax+1) = \alpha a x + \alpha \notin T$

so scalar multiplication is not closed

5 pts so T is not a subspace.

(could have shown + not closed or 0 not in T)

Note elements in T are of the form $ax+1$ and $bx+1$ (not) $ay+1$

(3) Is $\left\{ \begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\4\end{pmatrix}, \begin{pmatrix}1\\2\\6\end{pmatrix} \right\}$ a spanning set for $\mathbb{R}^3$?

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & x \\ 0 & 1 & 1 & 2 & y \\ 2 & 2 & 4 & 6 & z \end{array}\right) \xrightarrow{R_3-2R_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & x \\ 0 & 1 & 1 & 2 & y \\ 0 & 2 & 2 & 4 & z-2x \end{array}\right)$$

$$\xrightarrow{R_3-2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & x \\ 0 & 1 & 1 & 2 & y \\ 0 & 0 & 0 & 0 & z-2x-2y \end{array}\right)$$

10 pts Since $z-2x-2y$ is not necessarily 0 there is no linear combination of the given 4 vectors for which we can obtain a random $\begin{pmatrix}x\\y\\z\end{pmatrix}$ so the vectors do not span $\mathbb{R}^3$.

(4) Is $\left\{ \begin{pmatrix}2\\1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\4\end{pmatrix} \right\}$ a linearly independent set in $\mathbb{R}^4$?

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

10 pts

$$\xrightarrow{R_4 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow[2R_2+R_4]{R_2+R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\xrightarrow{-1/5 R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 9 \end{pmatrix} \xrightarrow{-9R_3+R_4} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are no free variables and there are 3 pivots in row echelon form the vectors are linearly independent.

(5) Suppose V is a vector space of dimension 5. Answer the following questions.

(a) If v_1, v_2, v_3, v_4, v_5 are linearly independent vectors of V, do they span V?

Yes since V is of dimension 5 and the vectors are linearly independent, they span.

(b) If v_1, v_2, v_3, v_4, v_5 are a spanning set for V, are they linearly independent in V?

Yes since V is of dimension 5 and the vectors span they are linearly independent.

4 pts

(c) Must a set of 6 vectors in V span V?

No they may be linearly dependent and span a subspace of dimension 4 or less.

(d) Can a set of 4 vectors in V be linearly independent?

Yes since $4<5$, they could be linearly independent.

(6) Find the coefficient vector of $1-2x+4x^3$ with respect to the basis $\{x^3, x^2, x, 1\}$ $= B$ of P_3.

5 pts

$$\left[\left(1-2x+4x^3\right)\right]_B = \begin{pmatrix} 4 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

(7) Determine if $L : P_2 \to P_3$ given by $L(p(x)) = xp(x) - x^2p'(x)$ is a linear transformation.

10 pts

$$\begin{aligned} L(p(x)+q(x)) &= x(p(x)+q(x)) - x^2(p(x)+q(x))' \\ &= xp(x) + xq(x) - x^2p'(x) - x^2q'(x) \\ &= xp(x) - x^2p'(x) + xq(x) - x^2q'(x) \\ &= L(p(x)) + L(q(x)) \end{aligned}$$

$$\begin{aligned} L(\alpha p(x)) &= x\alpha p(x) - x^2(\alpha(p(x)))' \\ &= \alpha\, xp(x) - \alpha x^2 p'(x) \\ &= \alpha L(p(x)) \end{aligned}$$

So L is a linear transformation.

(8) Consider the bases $E = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix} \right\}$ and $F = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$.

(a) Find the transition matrix from E to the standard basis.

5pts $U = \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}$ is the transition matrix from E to the standard basis.

(b) Find the transition matrix from the standard basis to F.

5pts $V = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is the transition matrix from F to the

$V^{-1} = \frac{1}{2-1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ is the transition matrix from the std to F.

(c) Find the transition matrix from E to F.

5pts $V^{-1}U = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ 7 & 12 \end{pmatrix}$ is the transition matrix from E to F.

(d) Given $[\overline{v}]_E = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, what is $[\overline{v}]_F$?

5pts $[\overline{v}]_F = \begin{pmatrix} -3 & -5 \\ 7 & 12 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -12 \\ 29 \end{pmatrix}$

(9) Given the matrix $A = \begin{pmatrix} 3 & 3 & 2 & -1 & -4 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$, find:

$R_1 \leftrightarrow R_3$

(a) A basis for the row space of A.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 3 & 3 & 2 & -1 & -4 \end{pmatrix} \xrightarrow[-3R_1+R_3]{-R_1+R_2} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -7 \end{pmatrix}$$

+5

$$\xrightarrow{-2R_2+R_3} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -9 \end{pmatrix}$$ so $\{(1\,1\,0\,0\,1), (0\,0\,1\,-1\,1), (0\,0\,0\,1\,-9)\}$ are a basis of the row space of A.

(b) A basis for the column space of A.

+3

Since there are pivots in the 1st 3rd and 4th columns then $\left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\}$ are a basis of the column space.

$R_3 + R_2$

(c) A basis for the nullspace of A.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & -9 \end{pmatrix}$$

$x_2 = s$, $x_5 = t$ are free variables

+5

$x_4 = 9t$, $x_3 = 8t$

$x_1 = -s - t$

so $\begin{pmatrix} -s-t \\ s \\ 8t \\ 9t \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 8 \\ 9 \\ 1 \end{pmatrix}$ and $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 8 \\ 9 \\ 1 \end{pmatrix} \right\}$ are a basis of $N(A)$.

(d) What is the rank of A, the nullity of A? Exhibit that the rank-nullity theorem holds for this matrix.

rank $A = 3$ nullity $A = 2$

+3

$3 + 2 = 5$ so rank nullity holds

(10) Find a matrix that represents the linear operator $L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_3\\x_1-x_2\\x_1+x_2\end{pmatrix}$ with respect to the basis $E=\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$.

$U=\begin{pmatrix}0&0&1\\1&1&0\\1&-1&0\end{pmatrix}$ is the transition matrix from E to the standard basis

$A=\begin{pmatrix}L(\bar{e}_1)&L(\bar{e}_2)&L(\bar{e}_3)\end{pmatrix}=\begin{pmatrix}0&0&1\\1&-1&0\\1&1&0\end{pmatrix}$ represents L with respect to the standard basis

10 pts

$$U^{-1}=\frac{1}{1\begin{vmatrix}1&1\\1&-1\end{vmatrix}}\begin{pmatrix}0&0&-2\\-1&-1&0\\-1&1&0\end{pmatrix}^T=\frac{1}{-2}\begin{pmatrix}0&-1&-1\\0&-1&1\\-2&0&0\end{pmatrix}$$

$$=\begin{pmatrix}0&1/2&1/2\\0&1/2&-1/2\\1&0&0\end{pmatrix}$$

The matrix representing L with respect to E is $U^{-1}AU=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\1&1&0\\1&-1&0\end{pmatrix}$

$$=\begin{pmatrix}0&0&1\\-1&-1&0\\1&-1&0\end{pmatrix}$$