Solutions

MATH 314–Linear Algebra
Quiz 7
Dr. Janet Vassilev

(1) Given $\overline{a} = (2, 0, 2)^T$ and $\overline{b} = (0, 1, 1)^T$. Find $\overline{a} \cdot \overline{b}$, $||\overline{a}||$, $||\overline{b}||$, and the angle between $\overline{a}$ and $\overline{b}$.

$$\overline{a} \cdot \overline{b} = 2 \cdot 0 + 0 \cdot 1 + 2 \cdot 1 = 2$$

$$||\overline{a}|| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$||\overline{b}|| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$$

$$\theta = \cos^{-1}\left(\frac{2}{2\sqrt{2} \cdot \sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \pi/3$$

(2) Find the vector projection of $(1, 1, 3)^T$ onto $(4, -1, 1)^T$.

$$\overline{p} = \frac{\langle (1,1,3)^T, (4,-1,1)^T \rangle}{||(4,-1,1)||^2} \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = \frac{6}{18} \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

(3) Are the subspaces $S = \{(a, a, a) \mid a \in \mathbb{R}\}$ and $T = \{(a + b, a - 2b, b - 2a) \mid a, b \in \mathbb{R}\}$ orthogonal subspaces? Justify your answer.

$$\begin{pmatrix} a \\ a \\ a \end{pmatrix} \cdot \begin{pmatrix} b+c \\ b-2c \\ c-2b \end{pmatrix} = a(b+c) + a(b-2c) + a(c-2b)$$

$$= 2ab - 2ab + 2ac - 2ac = 0$$

so yes they are orthogonal.

(4) For an $m \times n$ matrix A of rank n, express the least squares solution $\hat{x}$ of $A\overline{x} = \overline{b}$ in terms of a product of matrices times $\overline{b}$.

$$\hat{x} = (A^T A)^{-1} A^T \overline{b}$$

solutions

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Quiz 8
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Please use the back of the quiz if you need extra room. Label any problems on the back carefully.

(1) Using the inner product $< f, g >= \int_{-1}^{1} f(x)g(x)dx$ on the continuous functions $C[-1, 1]$, find $< x, x^3 >$, $||x||$ and $||x^3||$.

$$\langle x, x^3 \rangle = \int_{-1}^{1} x \cdot x^3 dx = \frac{1}{5}x^5 \Big|_{-1}^{1} = 2/5$$

$$||x|| = \sqrt{\int_{-1}^{1} x \cdot x dx} = \sqrt{\int_{-1}^{1} x^2 dx} = \sqrt{2/3}$$

$$||x^3|| = \sqrt{\int_{-1}^{1} x^3 \cdot x^3 dx} = \sqrt{2/7}$$

(2) Using the inner product $< A, B >= \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}b_{ij}$ on 2×2 matrices find an orthonormal basis for span$\left\{\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right\}$.

$$\left|\left| \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \right|\right| = \sqrt{1^2+2^2+2^2+0} = 3 \qquad u_1 = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

$$P_1 = \left\langle \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \right\rangle \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = \frac{6}{9}\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - 2/3\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 4/3 & -1/3 \\ -1/3 & 1 \end{pmatrix}$$

$$\left|\left| \begin{pmatrix} 4/3 & -1/3 \\ -1/3 & 1 \end{pmatrix} \right|\right| = \sqrt{\frac{16}{9} + \frac{1}{9} + \frac{1}{9} + 1} = \sqrt{\frac{27}{9}} = \sqrt{3}$$

$$u_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 4/3 & -1/3 \\ -1/3 & 1 \end{pmatrix}$$

so basis is $\left\{\frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 4/3 & -1/3 \\ -1/3 & 1 \end{pmatrix}\right\}$

(3) Suppose $\{u_1, u_2, u_3\}$ are an orthonormal basis of an inner product space V. Are the vectors $v = 2u_1 - u_2$ and $w = u_1 + 2u_2 + 3u_3$ orthogonal? What are the lengths of v and w?

$\langle v, w \rangle = 2 - 2 + 0 = 0$ so v & w are orthogonal

$||v|| = \sqrt{4+1} = \sqrt{5}$ $\qquad$ $||w|| = \sqrt{1+4+9} = \sqrt{14}$

Solutions

MATH 314–Linear Algebra
Quiz 9
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(1) Find the eigenvalues and corresponding eigenspaces of the matrix: $\begin{pmatrix} 2 & 2 & 1 \\ 10 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 10 & 3-\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 2-\lambda & 2 \\ 10 & 3-\lambda \end{vmatrix} = -\lambda\left[\lambda^2 - 5\lambda + 6 - 20\right]$$
$$= -\lambda(\lambda-7)(\lambda+2)$$

so the eigenvalues are $0, 7, -2$

$$0: \; N\begin{pmatrix} 2 & 2 & 1 \\ 10 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} = N\begin{pmatrix} 2 & 2 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 0 \end{pmatrix} = N\begin{pmatrix} 2 & 0 & 1/7 \\ 0 & -7 & -3 \\ 0 & 0 & 0 \end{pmatrix} = N\begin{pmatrix} 1 & 0 & 1/14 \\ 0 & 1 & 3/7 \\ 0 & 0 & 0 \end{pmatrix} = sp\left\{\begin{pmatrix} -1 \\ -6 \\ 14 \end{pmatrix}\right\}$$

$$7: \; N\begin{pmatrix} -5 & 2 & 1 \\ 10 & -4 & 2 \\ 0 & 0 & -7 \end{pmatrix} = N\begin{pmatrix} -5 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = sp\left\{\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}\right\}$$

$$-2: \; N\begin{pmatrix} 4 & 2 & 1 \\ 10 & 5 & 2 \\ 0 & 0 & 2 \end{pmatrix} = N\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = sp\left\{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\right\}$$

(2) Find a general solution to the system of differential equations:

$$y_1' = 4y_2$$
$$y_2' = y_1$$

$$Y' = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} Y \qquad \det\begin{pmatrix} -\lambda & 4 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 4 = (\lambda-2)(\lambda+2)$$

so eigenvalues are ± 2

$$2: \; N\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} = sp\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \qquad -2: \; N\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = sp\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$$

$$Y = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-2t}$$
$$= \begin{pmatrix} 2c_1 e^{2t} - 2c_2 e^{-2t} \\ c_1 e^{2t} + c_2 e^{-2t} \end{pmatrix}$$