

Homework 10
for MATH 421

From our text, please do problems 7.12 and 7.13. Then do the following problems.

- (A) A Galois extension K of a field F is abelian over F if $G(K : F)$ is an abelian group. Show that if K is abelian over F and $F \subseteq E \subseteq K$ is a subfield. Show that K is abelian over E and E is abelian over F .
- (B) Let $f \in K[x]$ be a polynomial of degree n , such that each irreducible factor is separable over K . Let L be the normal closure of f over K . Show that $|G(L : K)|$ divides $n!$.
- (C) A Galois extension is cyclic if its Galois group is cyclic. Show that if K is a cyclic extension of F , there exists exactly one subfield $F \subseteq E \subseteq K$ of degree d over F , for each divisor of $[K : F]$.
- (D) Give an example of fields F_1, F_2, F_3 with $\mathbb{Q} \subseteq F_1 \subseteq F_2 \subseteq F_3$, with $[F_3 : \mathbb{Q}] = 8$ and each field is Galois over all its subfields with the exception of F_2 which is not Galois.
- (E) Let $K = \mathbb{Q}(\sqrt[8]{2}, i)$ and let $F_1 = \mathbb{Q}(i)$, $F_2 = \mathbb{Q}(\sqrt{2})$ and $F_3 = \mathbb{Q}(i\sqrt{2})$. Prove that $G(K : F_1) \cong \mathbb{Z}_8$, $G(K : F_2) \cong D_4$ and $G(K : F_3) \cong Q$, where Q is the quaternions.