ABSTRACT ALGEBRA, MATH 520
HOMEWORK 1

JANET VASSILEV

(1) Let x be an element of a group G. Prove that $x^2 = e$ if and only if $o(x)$ is 1 or 2.
(2) If x and g are elements of a group G, prove that $o(x) = o(g^{-1}xg)$ and deduce that $o(ab) = o(ba)$ for all $a, b \in G$.
(3) Suppose x is an element of a group G and $o(x) = n = st$. Prove that $o(x^s) = t$.
(4) Suppose that $x^2 = e$ for all $x \in G$. Prove that G is abelian.
(5) Prove that any finite group of even order contains an element of order 2.
(6) Show that every element of $D_n =< r, s \mid r^n = s^2 = e, rs = sr^{-1} >$ which is not a rotation is of order 2. Prove that another presentation of D_n is given by $< a, b \mid a^2 = b^2 = (ab)^n = 1 >$ where $a = s$ and $b = sr$.
(7) Prove that the order of an element in S_n is the least common multiple of the lengths of the cycles in its cycle decomposition.
(8) Write out all the elements of $GL_2(\mathbb{Z}_2)$, determine the order of each element and show that $GL_2(\mathbb{Z}_2)$ is nonabelian.
(9) Prove that the order of $GL_2(\mathbb{Z}_p)$ is $p^4 - p^3 - p^2 + p$. (Your argument should be combinatorial in nature.)
(10) Find generators and relations for the group of quaternions.