

COMMUTATIVE ALGEBRA, MATH 530
HOMEWORK

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- (1) Let R be a commutative ring and $x \in R$. Show $x \in \text{rad}(R)$ if and only if $1 + rx$ is a unit for all $r \in R$, where $\text{rad}(R)$ is the Jacobson radical of R .
- (2) Let R be a ring and I be an ideal contained in $\text{nil}(R)$. Show that if $r + I$ is a unit in R/I then r is a unit of R .
- (3) Let R and S be commutative rings and $f : R \rightarrow S$ be a surjective homomorphism. Prove that $f(\text{rad}(R)) \subseteq \text{rad}(S)$. Give an example where the inclusion is strict.
- (4) Let $C = \{P_j \mid j \in \mathcal{J}\}$ be a collection of prime ideals in a commutative ring R . If C forms a chain then show $\bigcap_{j \in \mathcal{J}} P_j$ is a prime ideal. Additionally prove that for every proper ideal I , there are minimal elements among the primes containing I .
- (5) Let R be a commutative ring, I, P_1, \dots, P_n ideals of R with P_i prime for $3 \leq i \leq n$. Show that if I is not contained in any of the P_i then there exists an element $x \in I$ not contained in any P_i .
- (6) Let R be a commutative ring. Prove that if $R^n \cong R^m$ then $n = m$.
- (7) Let I be an ideal of a commutative ring R and M a finite R -module. Show $\sqrt{\text{ann}(M/IM)} = \sqrt{\text{ann}(M) + I}$.
- (8) Let R be a commutative ring. If $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of R -modules and K and N are finitely generated then M is finitely generated.
- (9) Let R be a commutative ring and M and N be R -submodules of an R -module L . Show that if $M + N$ and $M \cap N$ are finitely generated then so are M and N .
- (10) Let R be a commutative ring. If $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of R -modules and K and N are finitely presented then M is finitely presented.
- (11) Prove that if $0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_n \rightarrow 0$ is an exact sequence of R -modules of finite length, then $\sum_{i=1}^n (-1)^i \ell(M_i) = 0$.
- (12) Suppose $\{I_j\}_{1 \leq j \leq n}$ are ideals of a commutative ring R and $\bigcap_{j=1}^n I_j = (0)$. Show that if R/I_j are Noetherian, then R is Noetherian.
- (13) If R is a Noetherian ring, then any finitely generated module is finitely presented.
- (14) If R is Noetherian and $\phi \in \text{Hom}_R(R, R)$ is a surjective homomorphism then ϕ is injective.
- (15) Let (R, m) be a local ring with $m = (t)$ a principal ideal. Show that if $\bigcap_{n=1}^{\infty} m^n = (0)$ then R is Noetherian and every ideal is a power of m .
- (16) Show that an ideal I of a commutative ring R which contains m^n for some maximal ideal m and some natural number n is a primary ideal.
- (17) Show that $P^{(n)} = P^n R_P \cap R$ is a primary ideal with radical P . Consider $R = k[[x, y]]/(xy)$ and $P = (x)$, determine $P^{(n)}$ for all natural numbers n .
- (18) Let S be a multiplicative set of a commutative ring R , what is $\text{Spec } R_S$? Justify, your answer.
- (19) Prove that for any commutative ring R , every open cover of $\text{Spec } R$ has a finite subcover. We say that $\text{Spec } R$ is *quasi-compact*.

- (20) A topological space is Noetherian if it satisfies d.c.c. Show that if R is a Noetherian ring, then $\text{Spec } R$ is Noetherian.
- (21) Let S be the set of all nonzero divisors of a commutative R , prove that S is the largest subset of R satisfying $f : R \rightarrow R_S$ is injective and that every element of R_S is either a unit or a zero divisor.
- (22) For any R -module M define the torsion of M , $T(M) = \{x \in M \mid \text{ann } x \neq 0\}$. Show that for any multiplicative set S , $T(M)_S = T(M_S)$.
- (23) Prove that if $f : M \rightarrow N$ is an R -module homomorphism, show that $f(T(M)) \subseteq T(N)$ and if $0 \rightarrow K \rightarrow M \rightarrow N$ is exact then $0 \rightarrow T(K) \rightarrow T(M) \rightarrow T(N)$ is exact.
- (24) We say that M is torsionfree if $T(M) = 0$. Is $U_{TF} = \{P \in \text{Spec } R \mid M_P \text{ is torsionfree}\}$ an open set? Justify your answer.
- (25) Prove that for any prime of $k[x_1, \dots, x_n]$ satisfies $\text{ht } P + \text{coht } P = \dim R$.
- (26) Find the associated primes of the \mathbb{Z} -module $\mathbb{Z} \oplus \mathbb{Z}/120\mathbb{Z}$.
- (27) If M is a finite module over a Noetherian ring R and M_1 and M_2 are submodules of M with $M = M_1 + M_2$, is it true that $\text{Ass}(M) = \text{Ass}(M_1) \cup \text{Ass}(M_2)$?
- (28) If R is a Noetherian ring and x is a R -regular element which is not a unit, show that the $\text{Ass}(R/xR) = \text{Ass}(R/x^n)$ for any positive integer n .
- (29) Let I and J be ideals of a Noetherian ring R . Prove that if $JR_P \subseteq IR_P$ for all $P \in \text{Ass}(R/I)$ then $J \subseteq I$.
- (30) Give two primary decompositions of $(x^n, x^m y) \subseteq k[x, y]$ where $n > m$. Justify your answer.
- (31) Let R be a commutative ring and M and R -module. Show that M is a flat R -module if and only if M_P is a flat R_P module for all $P \in \text{Spec } R$.
- (32) Let R be a Noetherian ring and P and associated prime of R . Show there is an integer c such that $P \in \text{Ass}(R/I)$ for every ideal $I \subseteq P^c$.
- (33) Let $R \subseteq T$ be commutative rings with T integral over R . If \mathfrak{p} is a prime of R and T has only one prime P lying over \mathfrak{p} , then $T_P = T_{\mathfrak{p}}$.
- (34) Let $R \subseteq T$ be commutative rings with T integral over R . Show that $\dim R = \dim T$.
- (35) Let R be a domain with field of fractions K . We say $x \in K$ is almost integral over R if there exists $c \neq 0$ in R with $cx^n \in R$ for all $n > 0$. Prove that if $x \in K$ is integral over R then x is almost integral over R . Further prove that if R is Noetherian, then the converse holds.
- (36) Let $R \subseteq T$ be commutative rings with T integral over R . If P is a prime ideal of T and $\mathfrak{p} = P \cap R$, then $\text{ht } P \leq \text{ht } \mathfrak{p}$.
- (37) Let R, T be commutative rings with T an R -algebra such that the going down theorem holds between R and T . If P is a prime ideal of T and $\mathfrak{p} = P \cap R$, then $\text{ht } P \geq \text{ht } \mathfrak{p}$.
- (38) Prove that in a valuation ring any finitely generated ideal is principal.
- (39) If R is a valuation ring of dimension 1 and K is its field of fractions then there do not exist any intermediate rings between R and K . Conversely, show that if R is not a field and it is maximal among proper subrings, then R is a valuation ring of dimension 1.
- (40) Let R be a DVR and K its field of fractions. If L is a finite extension of K , then a valuation ring of L dominating R is a DVR.
- (41) Prove any ideal in a Dedekind ring can be generated by at most 2 elements.
- (42) Let R be the integral closure of \mathbb{Z} in $\mathbb{Q}[\sqrt{10}]$. Show that R is Dedekind but not a PID.
- (43) Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring and u a unit of R_0 . Show the map $T_u(x_0 + x_1 + \dots + x_n) = x_0 + x_1 u + \dots + x_n u^n$ is an automorphism of R , where $x_i \in R_i$. Show further that if R_0 contains an infinite field k then I is homogeneous if and only if $T_\alpha(I) = I$ for all $\alpha \in k$.
- (44) Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a \mathbb{Z} -graded ring. For an ideal I of R , let I^* denote the greatest homogeneous ideal contained in I . Show that if P is a prime, then P^* is a prime.

- (45) Let R be a \mathbb{Z} -graded integral domain. Let S be the multiplicative set consisting of all nonzero homogeneous elements of R . Show R_S is a graded ring and its degree zero component is a field.
- (46) Let R be a Noetherian ring with an embedded associated prime. If r is a nonzerodivisor with $\bigcap_{n \geq 0} r^n R = (0)$, then show $R/(r)$ also has an embedded associated prime.
- (47) Let (R, m) be Noetherian and set $G = \text{gr}_m(R)$. Show that if G is an integral domain, then so is R . Give an example of an integral domain so that G is not a domain. Justify your answer.
- (48) Let (R, m) be Noetherian and set $G = \text{gr}_m(R)$. For $r \in m^i \setminus m^{i+1}$, define r^* to be the image of r in m^i/m^{i+1} . Set $0^* = 0$. Prove the following:
- If $a^*b^* \neq 0$ then $(ab)^* = a^*b^*$.
 - If a^* and b^* have the same degree and $a^* + b^* \neq 0$, then $(a + b)^* = a^* + b^*$.
 - Let (R, m) be Noetherian and set $G = \text{gr}_m(R)$. Let $I \subseteq m$ be an ideal. Write I^* for the ideal generated by $r^* \in G$ for all $r \in I$. If $T = R/I$ and $n = m/I$, show $\text{gr}_n(T) = G/I^*$.
- (49) Let (R, m) be Noetherian and set $G = \text{gr}_m(R)$. Let $I = aR$. Show $I^* = a^*G$.
- (50) Let (R, m) be a regular local ring and K its field of fractions. For $0 \neq r \in R$, set $v(a) = i$ if $a \in m^i$ but $a \notin m^{i+1}$. Show v extends to an additive valuation of K .
- (51) Let T be the valuation ring of v from the previous problem. Prove that T is a DVR of K dominating A . Let x_1, \dots, x_d be a regular system of parameters of R , $S = R[x_2/x_1, \dots, x_d/x_1]$ and $P = x_1S$. Show P is a prime ideal of S and $S_P = T$.
- (52) Let (R, m) be as in the previous two problems. Show that if $0 \neq f \in m$ then $v(f) = e(m, R/(f))$.
- (53) Let (R, m) be a d -dimensional Noetherian local ring and x_1, \dots, x_d be a system of parameters of R with $\mathfrak{q} = (x_1, \dots, x_d)$. Set $\mathfrak{a} = (x_1, \dots, x_s)$. Let X be the set of all prime divisors of \mathfrak{a} with $\text{ht } \mathfrak{p} = s$ and $\text{coht } \mathfrak{p} = d - s$. Let M be a finite R -module. Use Lech's Lemma to prove $e(\mathfrak{q}, M) = \sum_{\mathfrak{p} \in X} e(\mathfrak{q} + \mathfrak{p}/\mathfrak{p}, R/\mathfrak{p})e(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}})$.
- (54) Let (R, m) be a Noetherian local ring, $M \neq 0$ a finite R -modules and $x_1, \dots, x_r \in m$ an M -sequence. Set $M' = M/(x_1, \dots, x_r)M$. Then $\dim M' = \dim M - r$.
- (55) Let R be a Noetherian local ring, M a finite R -module and P a prime ideal of R . Show $\text{depth}(P, M) \leq \text{depth}_{R_P}(M_P)$.
- (56) Let (R, m) and (S, n) be Noetherian local rings with $R \subseteq S$, $n \cap R = m$ and mS an n -primary ideal. Prove for any finite S -module M , $\text{depth}_S M = \text{depth}_R M$.
- (57) Let R be a Noetherian ring and M a finite R -module. The grade of an R -module M is defined to be

$$\text{grade} M := \inf\{i \mid \text{Ext}_R^i(M, R) \neq 0\}.$$

Justify that $\text{grade} M = \text{depth}(\text{ann} M, R)$ and $\text{grade} M \leq \text{proj dim} M$.

- (58) If R is a Noetherian ring and I is a proper ideal, we define I to be a perfect ideal if $\text{grade} I = \text{proj dim} R/I$. Prove that if I is a perfect ideal of grade n , then all its prime divisors have grade n .
- (59) Prove every zero dimensional ring is Cohen-Macaulay.
- (60) Prove every reduced 1-dimensional ring is Cohen-Macaulay.
- (61) Prove the a two dimensional normal ring is Cohen-Macaulay.
- (62) Let k be a field. For which n is $R_n = k[x^{n+1}, x^n y, x y^n, y^{n+1}]_P$, where $P = (x^{n+1}, x^n y, x y^n, y^{n+1})$ is R_n a Cohen-Macaulay ring. Justify your answer.
- (63) Prove that if R is a Cohen-Macaulay ring and x_1, \dots, x_r an R -sequence, then $R/(x_1^n, \dots, x_r^n)$ is Cohen-Macaulay for any n .
- (64) Let (R, m) be a Noetherian ring and x_1, \dots, x_r and R -sequence. Prove R is Gorenstein if and only if $R/(x_1, \dots, x_r)$ is Gorenstein.
- (65) Prove that if (R, m) is Gorenstein, then $R[x]$ is Gorenstein.

- (66) Let (R, m) be a local ring. Prove that $E_R(k)$ is a faithful R -module.
- (67) Determine $E_R(k)$ for $R = A/x^n A$ if A is a DVR.
- (68) If R is a DVR, describe R^\vee .
- (69) Look up the definition of the i th Bass number of a module with respect to a prime P . State the definition and interpret theorem 18.7 in the language of Bass numbers.
- (70) If R is Gorenstein, what are the i th Bass numbers of R with respect to any prime P ?