

Counting
for Janet Vassilev's Math 327 course

1 Counting Basics

Suppose A and B are sets each with a finite number of elements $|A| = n$ and $|B| = m$. If we want to determine the size of $A \cup B$ we need to know how many elements A and B share or the size of $A \cap B$. In fact

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For example, if $A = \{a_1, \dots, a_k, x_1, \dots, x_r\}$ and $B = \{b_1, \dots, b_\ell, x_1, \dots, x_r\}$, then

$$A \cup B = \{a_1, \dots, a_k, b_1, \dots, b_\ell, x_1, \dots, x_r\}$$

$$A \cap B = \{x_1, \dots, x_r\}$$

and

$$|A \cup B| = k + \ell + r = k + r + \ell + r - r = |A| + |B| - |A \cap B|.$$

Also we can determine the size of $A \setminus B$ and $B \setminus A$ given that we know the size of $A \cap B$. In particular $|A \setminus B| = |A| - |A \cap B|$ and $|B \setminus A| = |B| - |A \cap B|$.

Theorem 1.1 *If A and B are finite sets then*

1. $|A \cup B| = |A| + |B| - |A \cap B|$.
2. $|A \setminus B| = |A| - |A \cap B|$ and $|B \setminus A| = |B| - |A \cap B|$.

Example 1.2 *For example if we know that at a certain school there are 120 students that take French and 98 who take German and 23 students who take both French and German, then there are $120 + 98 - 23 = 195$ students studying French or German at that school.*

Example 1.3 *We can also use this theorem to determine how many integers between 1 and 100 are divisible by either 3 or 7. There are $\lfloor \frac{100}{3} \rfloor = 33$ integers divisible by 3 between 1 and 100. There are $\lfloor \frac{100}{7} \rfloor = 14$ divisible by 7 and $\lfloor \frac{100}{21} \rfloor = 4$ divisible by both 3 and 7. So there are $33 + 14 - 4 = 43$ integers between 1 and 100 divisible by either 3 or 7.*

Or maybe we want to know how many integers between 1 and 100 are divisible by 3 and not by 7. Using the above theorem we obtain that there are $33 - 4 = 29$ integers between 1 and 100 which are divisible by 3 but not 7.

Now consider three finite sets A , B and C .

$$\begin{aligned} |A \cup B \cup C| &= |A \cup (B \cup C)| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |(A \cap B) \cap (A \cap C)| \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \end{aligned}$$

Example 1.4 In Example 1.2, suppose we also knew that 300 students were studying Spanish and there are 35 students studying both French and Spanish, 10 students studying both Spanish and German but no students taking all three languages. How many students are studying one of the three languages French, German or Spanish? Using the above formula we see that there will be $300 + 120 + 98 - 23 - 35 - 10 = 450$ students studying one of these three languages.

We can generalize the formula for finding the size of the union of two or three sets as follows.

Theorem 1.5 Principle of Inclusion-Exclusion Let A_1, \dots, A_n be finite sets. Then

$$|A_1 \cup \dots \cup A_n| = \sum_{i=1}^n |A_i| - \sum_{i < j} |A_i \cap A_j| + \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n|.$$

Note that if the A_i are pairwise disjoint, then $|A_1 \cup \dots \cup A_n| = \sum_{i=1}^n |A_i|$ and this is what we call the **sum rule** for mutually exclusive events.

For example if you were trying to determine how many ways can you roll two die and obtain the sum 8? Let A_1 be when we roll 2 and 6, A_2 be when we roll 3 and 5 and A_3 be when we roll 4 and 4. Both A_1 and A_2 occur in two ways while A_3 only occurs one way so there are 5 ways to roll two die and obtain the sum 8.

Theorem 1.6 Multiplication Rule Let A_1, \dots, A_n be finite sets. Then

$$|A_1 \times \dots \times A_n| = \prod_{i=1}^n |A_i|.$$

For example at a restaurant, one has a choice of 4 salads, 10 entrees and 6 desserts, how many ways can you order a three course meal if you choose exactly one salad, one entree and one dessert? We use the multiplication rule and see that we have $4 \cdot 10 \cdot 6 = 240$ possible 3 course meals.

If at the same restaurant, you will only order two of the three courses, how many ways can you choose a two course meal. Here we will combine the sum rule and the multiplication rule. You can order either a salad and an entree, a salad and a dessert or an entree and a dessert. Hence, there are $4 \cdot 10 + 4 \cdot 6 + 10 \cdot 6 = 124$ ways to order 2 courses.

2 Permutations

Suppose 20 runners are participating in a Cross Country race. How many possible ways can the top 5 places be awarded? The first place can be chosen from among all 20 runners, but once the first place runner has been chosen, then there are only 19 competing for 2nd place, 18 for 3rd, 17 for 4th and 16 for 5th place. So there are $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 = 1860480$ ways to award the top 5 places.

Suppose you have three sculptures and 10 display cases in your house where you could put the sculptures. If each display case is to hold at most one sculpture, how many ways can you display your three sculptures? There are 10 ways to choose one display case for the first sculpture. Once this case has been chosen there are only 9 left to display the remaining

2 sculptures and once one of these is chosen there will only be 8 remaining cases to display the last sculpture so there are $10 \cdot 9 \cdot 8 = 720$ ways to display the sculptures.

In each of the above scenarios, we had n distinct objects and r distinct slots to place them in. For example with the runners we had 20 unique objects (the runners) and 5 unique slots (the places) that we wanted to place these runners. With the sculpture and the display cases, the objects are now the display cases and the slots are the sculptures. We define a **permutation** to be a set of distinct symbols which are arranged in order. An r -permutation of n symbols is a permutation of r of the n symbols. The number of r -permutations is $P(n, r) := n \cdot (n-1) \cdot (n-r+1)$. We will write $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$. We can express the number of r -permutations of n symbols by $P(n, r) = \frac{n!}{(n-r)!}$.

Example 2.1 *A man, a woman, a boy, a girl and a dog are lined up for a picture.*

1. *How many ways can they line up for the picture?*
2. *How many ways can they line up if the dog is in the middle?*
3. *How many ways can they line up if the man always follows the woman in the line up?*

Since there are 5 distinct living things lined up in the picture there are $5! = 120$ ways for them to line up.

If the dog is in the middle, then there are four people left to line up so there are $4! = 24$ ways for them to line up with the dog in the middle.

If the man always follows the woman, we can treat them as a block taking up two slots in the line. There are 4 ways to place the man and the woman in two adjacent spots where the man follows the woman. After the man and the woman have taken their spots, there are 3 spots left in the line where the boy, girl and the dog can line up so $3!$ ways. Using the multiplication rule we see there are $4 \cdot 3! = 4! = 24$ ways for them to line up this way too.