1   Combinations and Binomial Coefficients

Suppose now that we have n distinct objects and we want to select r of them without caring about the order. For example, we may have a red, a green, a blue and a yellow marble in a bag and we want to grab 2 of the marbles out of the bag. When we grab the marbles we don't do it in any particular order. Grabbing a blue and a yellow marble is indistinguishable from grabbing a yellow and a blue marble. There are 6 ways to select two marbles. Let R denote red, G denote green, B denote blue and Y denote yellow. The possibilities are RG, RB, RY, GB, GY and BY. This is an example of the set of 2 combinations of 4 which we will define below.

Definition 1.1 *An r-combination of n distinct objects is any collection of r of the objects. The number of r-combinations of n objects is denoted either $C(n, r)$ or $\binom{n}{r}$ and is $\dfrac{n!}{r!(n-r)!}$.*

Example 1.2 *Suppose a 5 member committee is to be chosen from a group of 4 women and 6 men.*

   1. How many committees can be chosen?

   2. How many committees can be chosen if the committee has exactly 2 women.

   3. How many committees can be chosen if has at most two women.

   (1) Since there are 10 people, there are $C(10, 5) = \dfrac{10!}{5!5!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2} = 252$.
   (2) If exactly two committee members are women, then there are 4 women to choose the 2 women from and 6 men to choose the 3 men from. So there are $C(4, 2) = 6$ ways to choose the women on the committee and $C(6, 3) = 20$ ways to choose the men. Now using the multiplication principle, there will be $6 \cdot 20 = 120$ ways of choosing a committee with 2 women and 3 men.
   (3) If at most two women are on the committee, then either there are no women, one woman or two women. These scenario's are mutually exclusive so the sets of committees with no women, one woman and two women will be pairwise disjoint. So using the addition principle we can add the number of committees with no women to the number with one woman to the number with two women to find our answer. There are $C(6, 5) = 6$ ways of choosing a committee with no women. There are $C(7, 4)C(4, 1) = 35 \cdot 4 = 140$ ways of choosing a committee with one woman. There are 120 ways of choosing a committee with two women as we did in (2) above. So there are $6 + 140 + 120 = 266$ ways of choosing a committee with at most two women.

Example 1.3 *Suppose we have 6 ceramic sculptures which could be on display in any of 13 display cases. Assuming that each display case can hold at most one sculpture. How many ways can we display 4 of these sculpture in the display cases?*
   We first have to choose which of the four sculptures will be on display. There are $\binom{6}{4} = 15$ ways of choosing the 4 sculptures. Now that we have chosen the sculptures, we need to determine the 4 locations for the sculptures. Since there are 13 cases, there are $P(13, 4) = 13 \cdot 12 \cdot 11 \cdot 10 = 17,160$ ways of displaying the 4 chosen sculptures. So there are $15 \cdot 17160 = 257,400$ ways of displaying any of the 4 sculptures.

The r-combinations of n are used heavily in the binomial theorem. Consider $(x+y)(x+y)(x+y)(x+y) = (x+y)^4$. In expanding this expression as a sum of terms, we need to determine how many copies of x^4, x^3y, x^2y^2, xy^3 and y^4 there are. There will be only one copy of x^4 since from each of the four copies of $x+y$ in the product we will choose the x or 0 copies of y. There will be $4 = \binom{4}{1}$ copies of x^3y since we can choose one y from each of the four copies of $x+y$. Similarly there will be $6 = \binom{4}{2}$ copies of x^2y^2, $4 = \binom{4}{3}$ copies of xy^3 and $1 = \binom{4}{4}$ copy of y^4. So $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

The binomial theorem states the following:

Theorem 1.4 *For any nonnegative integer* $(x+y)^n = \sum\limits_{i=0}^{n} \binom{n}{i} x^{n-i}y^i$.

Before proving the theorem, notice that the binomial coefficients appearing in the sum above correspond to the coefficients in the nth row of Pascal's triangle.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 6 & & 1 \\
\end{array}
$$

This triangle continues infinitely downward. Notice that every entry is the sum of the two entries directly above it.

Proposition 1.5 *For all* $n \geq 2$ *and all* $1 \leq r \leq n$, $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

Proof:

$$
\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(n-1-(r-1))!(r-1)!} + \frac{(n-1)!}{(n-1-r)!r!} \\
&= \frac{(n-1)!}{(n-r))!(r-1)!} + \frac{(n-1)!}{(n-1-r)!r!} \\
&= \frac{r(n-1)!}{(n-r)!r!} + \frac{(n-r)(n-1)!}{(n-r)!r!} \\
&= \frac{n(n-1)!}{(n-r)!r!} \\
&= \binom{n}{r}
\end{aligned}
$$

Now using this proposition we can prove the binomial theorem.

Proof of the binomial theorem: Note that for $n = 1$, $(x+y)^1 = x+y = \binom{1}{0}x + \binom{1}{1}y$. Assume that $(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$ for some $n \geq 2$. Multiply both sides by $(x+y)$ to obtain

$$
\begin{aligned}
(x+y)^{n+1} &= (x+y)\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i \\
&= \sum_{i=0}^{n}\binom{n}{i}x^{n+1-i}y^i + \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^{i+1} \\
&= \sum_{i=0}^{n}\binom{n}{i}x^{n+1-i}y^i + \sum_{i=1}^{n+1}\binom{n}{i-1}x^{n-i}y^i \\
&= \binom{n}{0}x^{n+1} + \sum_{i=1}^{n}\left(\binom{(}{n}i + \binom{n}{i-1}\right)x^{n+1-i}y^i + \binom{n}{n}y^{n+1} \\
&= \binom{n+1}{0}x^{n+1} + \sum_{i=1}^{n}\binom{n+1}{i}x^{n+1-i}y^i + \binom{n+1}{n+1}y^{n+1} \\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}x^{n+1-i}y^i
\end{aligned}
$$

where the second to last equality came from the above proposition and the second equality is a consequence of re-indexing. Thus by induction, we have proved the binomial theorem.

As a consequence of the theorem, we can see that each row of Pascal's Triangle sums to 2^n since $2^n = (1+1)^n = \sum_{i=0}^{n}\binom{n}{i}$.

2 Repetitions

Suppose now that we allow some repetition of our objects. For example, suppose we have three identical white marbles and we want to place them among 10 distinct boxes. If at most one marble goes in each box, we know how to count this already. We just need to choose three boxes from the 10 or we have $\binom{10}{3} = 120$ ways of placing the marbles. Now suppose we can put any number of marbles into a box. We can either put at most one marble in a box, or we could put 2 in one box and one in another box, or we could put all three in one box. There are $2!\binom{10}{2} = 90$ ways of placing 2 marbles in one box and one in another. The reason why we multiplied by 2! is because once we have chosen the two boxes, we can either place 2 in the first box or two in the second box. There are 10 ways of placing all three in the 10 boxes. Using our addition principle, we find that there are 220 ways of placing the 3 marbles in the 10 boxes. Notice that $\binom{10+3-1}{3} = \dfrac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220$.

If we try to put 9 identical marbles into 4 distinct boxes, this is much harder to analyze on a case by case basis but actually there are $\binom{4+9-1}{9} = 220$ ways of placing the 9 marbles in the 4 boxes. We will use the following proposition for counting the placement of identical objects in distinct boxes.

Proposition 2.1 *There are $\binom{n+r-1}{r}$ ways of placing r identical objects into n distinct boxes.*

Suppose we have 3 identical red marbles and 2 identical blue marbles. How many ways can we place them in 7 distinct boxes? There are $\binom{7+3-1}{3} = 84$ ways to place the red marbles in the boxes and $\binom{7+2-1}{2} = 28$ ways to place the blue marbles. Using the multiplication principle, we have $84 \cdot 28 = 2352$ ways of placing the marbles.

We can also use the above proposition to determine the nonnegative integer solutions to $x_1 + x_2 \cdots + x_n = r$. We can think of $r = 1 + 1 + \cdot + 1$ and the solutions to this equation will be the placements of 1's into the "box" for each variable. So the number of solutions is $\binom{n+r-1}{r}$.

We also need to be careful when we have permutations involving some objects which are not distinct. For example, suppose we want to determine the number of anagrams of CHEESE. The E's are not distinct. First pretend the E's are distinct. For example, on is E, the other e and the last is ϵ. There are $6! = 720$ anagrams with different representations for each E. However, since they were all the same after all, we will divide by 3! which is the number of ways that we can order the three "distinct" E's. So there are $\dfrac{6!}{3!} = 120$ anagrams of CHEESE.

If we want to place 4 red marbles, 3 blue marbles and 2 green marbles in 9 boxes so that each box has at most one marble, we can use the multiplication principle to find that there are $\binom{9}{4}\binom{5}{3}\binom{2}{2} = \dfrac{9!}{4!3!2!}$. Notice the similarity to the last example.