Abstract Algebra
Homework 3

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1. Let G be a group and N_i be a collection of normal subgroups. Then $\bigcap_i N_i$ is normal in G.

2. Suppose $S \subseteq Z(G)$ is a subgroup of $Z(G)$. Show $S \triangleleft G$. Also show if G/S is cyclic, then G is abelian.

3. If H is a subgroup of G and $[G : H] = 2$, show that $H \triangleleft G$. Show the corresponding statement is false if 2 is replaced by 3.

4. Show that the normalizer, $N_G(H)$, of H is the largest subgroup of G containing H for which $H \triangleleft G$.

5. Let $M = \bigcap_{x \in G} xHx^{-1}$. Show that $M \triangleleft G$ and M is the largest normal subgroup of G which is contained in H.

6. We say that H is *characteristic* in G written $H \triangleleft\triangleleft G$ if for any $\sigma \in \mathrm{Aut}(G)$, $\sigma|_H \in \mathrm{Aut}(H)$. Show if $H \triangleleft\triangleleft G$ then $H \triangleleft G$. Also show that if $K \triangleleft\triangleleft H \triangleleft G$ then $K \triangleleft G$.

7. Show $Z(G) \triangleleft\triangleleft G$.

8. Suppose $N \triangleleft G$, and $[G : N]$ is finite. Suppose H is a finite subgroup and $|H|$ and $[G : N]$ are relatively prime. Show that $H \subseteq N$.

9. Suppose $N \triangleleft G$ and $|N|$ finite. Suppose H is a subgroup of finite index and $|N|$ and $[G : H]$ are relatively prime. Show that $N \subseteq H$.

10. Suppose G is a group. A proper subgroup H of G is *maximal* if no proper subgroup of G contains H. Let $\phi(G) = \bigcap_{M \text{maximal}} M$. Show that $\phi(G) \triangleleft\triangleleft G$. Let S be a subset of G and $x \in \phi(G)$. If $\langle S, x \rangle = G$, show that $\langle S \rangle = G$.