Abstract Algebra
Homework 4

Dr. Janet Vassilev

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1. Let $\mathbb{Q}$ be the additive group of rational numbers and $\mathbb{Z}$ the additive group of integers. Let $M = \mathbb{Q}/\mathbb{Z}$. Let p be a prime. Let G be the subgroup of M generated by all cosets containing fractions of the form a/p^n, $n = 1, 2, 3, \ldots, a$ and integer, $(a, p) = 1$. Show that every proper subgroup of G is cyclic of order p^n for some positive integer n. Conclude that G has no maximal subgroup.

2. Let p be a prime. Show that every group of order p^2 is abelian.

3. Let G_1 be a group of order 8 generated by two elements x and y subject to the relations:
$$x^4 = e,\ y^2 = e,\ yxy^{-1} = x^{-1}$$
Show that $\langle x^2 \rangle$ is a normal subgroup of order 2, but $\langle y \rangle$ is a subgroup of order 2 that is not normal. This shows that the existence of normal subgroups of all possible orders does not imply all subgroups are normal.

4. Let G_2 be another group of order 8 generated by two elements x and y subject to the relations:
$$x^4 = e,\ y^4 = e,\ x^2 = y^2,\ yxy^{-1} = x^{-1}$$
Note that one difference between G_1 and G_2 is that y has order 2 in one and order 4 in the other. Show that every subgroup of G_2 is normal. Thus, all subgroups normal does not make a group abelian. Prove, however, that G_1 and G_2 are the only non-abelian groups of order 8 up to isomorphism.

5. Let F be a field and G the multiplicative group of non-zero elements of F. Show that every finite subgroup of G is cyclic. (Hint: Reduce to the case where the finite subgroup is a p-group and argue that the polynomial $x^n - 1$ cannot have more than n roots in a field F.)

6. Suppose p and q are primes and F is a field of integers modulo q. Note that the multiplicative group of F has order $q - 1$. Show that the congruence $i^p \equiv 1 (\text{mod } q)$ has no non-trivial solution if q is not congruent to 1 mod p, but does have non-trivial solutions when q is congruent to 1 mod p. In the

latter case, show that all non-trivial solutions are of the form u^j where u is an integer which is not 1 mod q, and $1 \leq j \leq p-1$.

7. Using 6) and its context, show that if $q \equiv 1 (\text{mod} p)$, there is a unique (up to isomorphism) non-abelian group of order pq. (Hint: Use 6 to find the candidates, then show they are isomorphic.)

8. Let G be a finite group having a proper subgroup which contains all other proper subgroups. Prover that G is cyclic of prime power order.

9. Show there is no simple group of order 484 or 36.

10. Let P be a non-abelian p-group in which all subgroups are normal. Show that $p = 2$.