Abstract Algebra
Homework 6

Dr. Janet Vassilev

September 26, 2012

1. Suppose G is a finite group and H a proper subgroup. Show that there is an element of G that has no conjugate in H. (Hint: if $q = [G : H]$, then the conjugates of H account for at most $q(|H| - 1) + 1$ elements of G.

2. Let G be a finite group acting transitively on a set Ω. Then either there is an element of G acting without fixed points on Ω, or Ω has one element.

3. Let G be a finite group which acts transitively on a set Ω. Prove or give a counterexample to the following: If $|\Omega| = n$ then G has an element of order n.

4. Let $\{e\} \neq H \triangleleft G \subseteq S_p$ for p a prime. Prove that if G is transitive on $O = \{1, 2, \ldots, p\}$, then so is H. Conclude that p divides $|H|$. (Hint: show that $H \nsubseteq \mathrm{Stab}_G(a)$ for $a \in O$. Now look at the subgroup of G generated by H and $\mathrm{Stab}_G(a)$ for some $a \in O$.

5. Let G be a group of order $2n$ where n is odd. Cayley's theorem says that G can be considered as a subgroup of S_{2n}. Show in this embedding that elements of odd order in G end up in A_{2n}. Conclude that the elements of odd order in G form a subgroup and that there are no simple groups of order $2n$, n odd.

6. Show that A_n is generated by its 3-cycles for all $n > 2$.

7. Show that S_n is generated by the transpositions $(12), (23), \ldots, (n - 1n)$.

8. In S_n, let $H = \{\sigma \mid \sigma(1) = 1\}$. Is H a maximal subgroup of S_n?

9. Let G be a finite group and H a subgroup of index n. If H contains no normal subgroup of G besides $\{e\}$, then G is isomorphic to a subgroup of S_n. Use this to show that $D_4 \subseteq S_4$, but $Q \nsubseteq S_n$ unless $n \geq 8$.

10. Show any simple group of order 60 is isomorphic to A_5. The hunt should be for subgroups of order 12.