Abstract Algebra
Homework 9

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October 24, 2012

1. Let R be a ring and M a left R-module. Show that if M is cyclic then $M \cong R/I$ for some left ideal I.
2. Show that $M_n(F)$ is a simple ring for any field F.
3. An R-module M is Noetherian if M and all its submodules are finitely generated. Show the following are equivalent:
   (a) M is Noetherian.
   (b) Every ascending chain of R-submodules $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$ terminates with $M_k = M_{k+i}$ for every $i \geq 0$.
   (c) Every collection of submodules M has a largest element with respect to inclusion.
4. Let R be a commutative ring with 1. Let M be an ideal. Show that M is a maximal if and only if R/M is a field.
5. Let R be a ring. R is left-Noetherian if every left ideal of R is finitely generated. Suppose R is such and M is a finitely generated R-module. Show that M is a Noetherian R-module.
6. An R-module M is called irreducible if it has no non-trivial R-submodules except the obvious ones $\{0\}$ and R. Suppose N is an R-module and $M_1, M_2, \ldots, M_n$ are irreducible submodules so that $N = M_1 + M_2 + \cdots M_n$. Show that N is a direct sum of some of the M_i.
7. Suppose $0 \to N \xrightarrow{f} M \xrightarrow{g} E \to 0$ is an exact sequence of R-modules. Show that any two of N, M and E is Noetherian if and only if the third one is also.
8. Suppose R is a commutative ring with 1. The Jacobson radical $J(R)$ is defined to be the intersection of all maximal ideals of R. Let $x \in R$. Prove that $x \in J(R)$ if and only if $1 - xy$ is invertible for all $y \in R$.
9. Let R be a commutative ring with 1. Show that a and b are associates if and only if $a|b$ and $b|a$ if and only if $(a) = (b)$.

10. Show that $\mathbb{Z}/(mn) \cong \mathbb{Z}/(m) \times \mathbb{Z}/(n)$ if $(m, n) = 1$. Conclude $U(mn) \cong U(m) \times U(n)$ where $U(m)$ is the group of units of $Z/(m)$.