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Star, semistar and standard operations: A case study

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A B S T R A C T

The theory of star and semistar operations has been developed mostly over domains. Recently, Epstein defined star and semistar operations over more general commutative rings. We will compare some of the known results over domains with the non-domain setting. We conclude with a classification of the standard closures on the nodal curve.

Both star operations and prime operations were introduced by Krull in his 1935 book Idealtheorie [12] without any distinction between the two. In fact, the symbol he used for such operations was the prime symbol ($'$). A year later Krull published another work on star/prime operations [13] where he switched the symbol to a star $*$. Krull worked in the domain setting where these two operations are in fact identical. Gilmer picked up the star notation and published it in his book Multiplicative Ideal Theory [6] and subsequently many results appeared on star operations over domains. However, the star symbol at some point morphed into $\star$ which is the more common symbol used to represent a star operation today. Petro in [21] used Krull's original notation $\prime$ and generalized the notion of prime operations to semiprime operations. In a quite dissimilar way, star operations

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were generalized to semistar operations by Matsuda and Okabe [15]. Although work on prime and semiprime operations has been studied in the more general commutative ring setting in papers such as [22], until recently star operations and semistar operations were only in the domain setting. Epstein introduced star operations in [3] and semistar operations in [4] over a more general commutative ring. Surprisingly, many facts that are true in the domain setting are no longer the case.

In Section 1, we will review closure operations on the set of ideals, the set of fractional ideals and the set of R-submodules of the total ring of fractions of a commutative ring. We highlight some interesting examples which lead to some new definitions of closure operations. In Section 2, we review the history about counting star and semistar operations in the domain setting and prove a result about counting closure operations on rings with a finite number of ideals. In Section 3, we prove our main result which is to exhibit that there are precisely 24 standard closures on the nodal curve $k[[x,y]]/(xy)$. Due to Epstein's result [4] which exhibits a bijection between finite type semistar operations and standard closures, we conclude that there are 24 finite type semistar operations on $k[[x,y]]/(xy)$.

1. Closure operations

Let R be a commutative ring with unity. We will denote the units of R by $U(R)$ and the zero divisors of R by $Z(R)$. Throughout $R^\times$ will denote the nonzero elements of R. We denote the set of ideals of R by $\mathcal{I}(R)$ and the set of finitely generated ideals by $\mathcal{I}_f(R)$. Let Q be the total ring of fractions of R. Recall that a fractional ideal of R is an R-submodule A of Q satisfying the property that there exists a regular element $x \in R$ such that $xA \subseteq R$. We denote the R-submodules of Q by $\overline{\mathcal{F}}(R)$, the fractional ideals of R by $\mathcal{F}(R)$ and the finitely generated fractional ideals by $\mathcal{F}_f(R)$. A **closure operation** on the set of ideals of a commutative ring R (respectively the set of fractional ideals of R or the R-submodules of Q) is a function $c : \mathcal{I}(R) \to \mathcal{I}(R)$ (respectively $c : \mathcal{F}(R) \to \mathcal{F}(R)$ or $c : \overline{\mathcal{F}}(R) \to \overline{\mathcal{F}}(R)$) satisfying:

- (Extension) $I \subseteq I^c$ for all $I \in \mathcal{I}(R)$ ($I \in \mathcal{F}(R)$ or $I \in \overline{\mathcal{F}}(R)$).
- (Order Preservation) If $I \subseteq J$ then $I^c \subseteq J^c$.
- (Idempotence) $(I^c)^c = I^c$ for all $I \in \mathcal{I}(R)$ ($I \in \mathcal{F}(R)$ or $I \in \overline{\mathcal{F}}(R)$).

A closure operation c on the set of ideals of R is called **semiprime** if in addition any of the equivalent conditions hold:

- For all ideals $I, J \in \mathcal{I}(R)$, $IJ^c \subseteq (IJ)^c$,
- For all ideals $I, J \in \mathcal{I}(R)$, $I^cJ^c \subseteq (IJ)^c$,
- For all $x \in R$ and all $I \in \mathcal{I}(R)$, $xI^c \subseteq (xI)^c$.

Epstein defined a closure operation c on the set of ideals of R to be **weakly prime** if for all non-zero divisors $u \in R$, $uI^c \subseteq (uI)^c$ in [4]. Note every semiprime operation must be weakly prime by the third bullet above. However, there are weakly prime closures which are not semiprime. The following example was suggested by Epstein [5].

Example 1.1. Let $R = k[x]/(x^2)$ with k a field. All the elements of R can be written in the form $a + bx$ for some $a, b \in k$. As long as $a \neq 0$, $a + bx$ is a unit. Moreover, R is principal ideal ring because $k[x]$ is a PID. Hence, the only ideals of R are $(0), (x)$ and R. Define a closure operation c on the ideals of R as follows $(0)^c = (0)$ and $(x)^c = R^c = R$. Note that the regular elements of R are of the form $a + bx$ for any nonzero $a \in k$. $(a+bx)I = I$ for all ideals I in R. Thus, c is weakly prime. However, c is not semiprime since $x(x)^c = (x) \nsubseteq (x^2)^c = (0)$.

In [3] Epstein defined a closure operation c on the set of ideals to be **star** if for all regular elements $u \in R$ and all $I \in \mathcal{I}(R)$, $(uI)^c = uI^c$.

Example 1.2. Let $R = k[[x, y]]/(xy)$. The non-zero divisors are the units $a + \sum_{i\geq 1} b_i x^i + \sum_{j\geq 1} c_j y^j$ for a nonzero and the elements of the form $\sum_{i\geq 1} b_i x^i + \sum_{j\geq 1} c_j y^j$ where at least one $b_i \neq 0$ and at least one $c_j \neq 0$. It is clear to see the ideals of R are of one of the following forms:

- (0) or R,
- (x^n) for some $n \in \mathbb{N}$,
- (y^n) for some $n \in \mathbb{N}$,
- (x^m, y^n) for some $m, n \in \mathbb{N}$,
- $(x^m + ay^n)$ for some $m, n \in \mathbb{N}$ and $a \in k^\times$.

There are closure operations on the set of ideals of R which are not star operations. For example, define c as follows:

$$I^c = \begin{cases} I \text{ for } I \neq (x, y) \\ R \text{ for } I = (x, y). \end{cases}$$

c is not star because $(x + y)(x, y)^c = (x + y) \neq (x^2, y^2) = [(x + y)(x, y)]^c$.

Although there are many closure operations on $R = k[[x, y]]/(xy)$ which are not star, we can show that there is only one star operation.

Proposition 1.3. *Let $R = k[[x, y]]/(xy)$. The only star operation on the set of ideals of R is the identity.*

Proof. Since R is the largest ideal, for any star operations $\star$, $R^\star = R$. Thus for all regular elements r in R, $rR = (rR)^\star$. So all the ideals of the form $(x^m + ay^n)$ must be $\star$-closed.

Suppose $(x, y)^\star = R$. For all $m, n \in \mathbb{N}$, the equality

$$(x^m + y^n)(x, y)^\star = ((x^m + y^n)(x, y))^\star$$

implies $(x^{m+1}, y^{n+1})^\star = (x^m + y^n)$. Since $(x, y^2)^\star \subseteq (x, y)^\star = R$ either $(x, y^2)^\star = (x, y^2)$ or $(x, y^2)^\star = R$. Suppose first that $(x, y^2)^\star = (x, y^2)$. Then

$$(x + y)(x, y^2)^\star = ((x + y)(x, y^2))^\star$$

implies that $(x^2, y^3) = (x^2, y^3)^\star = (x + y^2)$ which is a contradiction. Now suppose $(x, y^2)^\star = R$. We obtain another contradiction through the equalities

$$(x + y) = (x + y)(x, y^2)^\star = ((x + y)(x, y^2))^\star = (x + y^2).$$

Hence, it must be the case that $(x, y)^\star = (x, y)$, and we conclude that

$$(x^{m+1}, y^{n+1}) = (x^m + y^n)(x, y)^\star = ((x^m + y^n)(x, y))^\star = (x^{m+1}, y^{n+1})^\star.$$

Thus $(x^m, y^n)^\star = (x^m, y^n)$ for all natural numbers $m, n > 1$. Suppose $(x, y^n)^\star = (x, y^k)$ for $1 \leq k \leq n$. Since

$$(x^2, y^{k+1}) = (x + y)(x, y^n)^\star = ((x + y)(x, y^n))^\star = (x^2, y^{n+1})$$

then it must be the case that $k = n$. Similarly, $(x^n, y)^\star = (x^n, y)$.

Since $(x^m)^\star \subseteq \cap_{n=1}^\infty (x^m, y^n)^\star = \cap_{n=1}^\infty (x^m, y^n) = (x^m)$ we have $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similarly $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Finally since $(0)^\star \subseteq \cap_{m=1}^\infty (x^m)^\star = \cap_{m=1}^\infty (x^m) = (0)$ we have $(0)^\star = (0)$. □

Remark 1.4. In the case that the base field is not $\mathbb{F}_2$ we can streamline the above proof by replacing the second paragraph above by:

Note that

$$(x^m + ay^n) \subseteq (x^m, y^n) \subseteq (x^{m-1} + by^{n-1})$$

for all nonzero $a, b \in k$, and

$$\bigcap_{b \in k^\times} (x^{m-1} + by^{n-1}) = (x^m, y^n).$$

Putting this fact together with order preservation we see that $(x^m, y^n)^\star = (x^m, y^n)$.

A closure operation on the R-submodules of Q is termed **divisible** if for all regular elements u of Q and all R-submodules A of Q, $(uA)^c = uA^c$. A **prime operation** is a semiprime star operation. The terms semiprime operation and prime operation have typically been defined on multiplicatively closed sets of ideals contained in a commutative ring, for multiplicatively closed sets of filtrations of ideals in a ring or for multiplicatively closed sets of R-submodules of an R-module R. For some examples, see [23,21,24,19,20], or [7]. However, we can also define these notions on the set of fractional ideals of a commutative ring R using the same axioms as above. In fact, the current author analyzed the semiprime operations on the set of fractional ideals of a DVR in [25].

Traditionally star operations were defined on the nonzero fractional ideals of a domain D. Epstein's definition of a star operation on the ideals of a ring was motivated by Krull's original definition of a star operation [12] and [13]. Note if D is a domain, all nonzero elements of D are regular and D is an ideal. A quick consequence is that $D^\star = D$. However, when one moves to the setting of fractional ideals, it may be the case that $D \subsetneq D^\star$. Thus in the definition of star operation on the set of fractional ideals we need the extra assumption that $D = D^\star$. Matsuda and Okabe [15] not only relaxed this assumption in their definition of semistar operation but also defined a **semistar operation** to be a closure operation on the nonzero R-submodules of the fraction field of a domain D which is divisible. We should mention here that originally star operations were defined on the nonzero fractional ideals of a domain D and semistar operations were defined on the nonzero R-submodules of the fraction field of a domain D. Epstein noted in [4] that one can define a semistar operation on all the R-submodules of the total ring of fractions of any commutative ring R as a divisible closure satisfying $(0)^\star = (0)$. We note that a star operation can also be defined on the set of all the fractional ideals of a commutative ring R. However, to be consistent with star operations defined on domains we will define a star operation $\star$ on the set of fractional ideals of R as follows:

Definition 1.5. Let R be a commutative ring. A closure operation $\star$ on $\mathcal{F}(R)$ is a star operation if $\star$ is a divisible closure which also satisfies $R^\star = R$.

Note, that any star operation $\star$ on the ideals or fractional ideals of a domain is in fact semiprime. Let $r \in IJ^\star$. Suppose $S = \{i_\lambda \mid \lambda \in \Lambda\}$ is a generating set for I with all $i_\lambda \neq 0$. Then for some $r_1, \dots, r_n \in J^\star$ and $i_1, \dots, i_n \in S$, $r = \sum_{j=1}^{n} r_j i_j$. Thus

$$r \in \sum_{j=1}^{n} i_j J^\star = \sum_{j=1}^{n} (i_j J)^\star \subseteq (IJ)^\star.$$

Hence, any star operation on the ideals or fractional ideals of a domain is a prime operation.

When R is a commutative ring and Q is its total ring of fractions, one defines the fractional ideal $I^{-1} = \{x \in Q \mid xI \subseteq R\}$. One of the most important star operations on the set of fractional ideals in the domain setting is the v-operation which was defined to be $I_v = (I^{-1})^{-1}$ for all fractional ideals I of R. The v-operation, although mainly studied in the domain setting, was looked at briefly in [11]. In fact, Huckaba shows that if I is a fractional ideal with a regular element then I is the intersection of all cyclic R-submodules of Q containing I. This is, in fact, how Epstein defines I_v in [3] for any ideal I of R. However, Epstein's definition does not agree with Huckaba's when there are zero divisors. In fact, in [11] Huckaba mentions that even for fractional ideals which contain a regular element $(I^{-1})^{-1}$ may not be the intersection of cyclic R-submodules of Q containing I.

One condition of a star operation $\star$ that is lost when there are zero divisors is the condition that $(x)^\star = (x)$ or $(xI)^\star = xI^\star$ for all elements $x \in Q$ and all ideals (or fractional ideals) of R. Huckaba actually defined star operations on the fractional ideals of some rings with zero divisors in [11] to be prime operations on $\mathcal{F}(R)$.

Example 1.6. Let $R = k[x, y]/(x, y)^2$. Note that $Q = R$ since R is Artinian. We define a star operation $\star$ on the ideals (also the fractional ideals) of R by $(a)^\star = (a)$ for all principal ideals (a) of R and $(x, y)^\star = R = R^\star$. Note that $(x)(x, y) \subseteq (0)$ in R. Thus $((x)(x, y))^\star = (0)$. However, $(x)(x, y)^\star = (x)R = (x)$. This example is motivated by [3, Proposition 4.1.7(iii)]. Epstein noted that this star operation is an example of a star operation which is not semiprime. Recall, this couldn't happen for domains.

This prompts the following new definitions.

Definition 1.7. Let R be a commutative ring. Let $\star$ be a star operation defined on $\mathcal{I}(R)$ (or $\mathcal{F}(R)$). We say $\star$ is a **principally closed star operation** if for all $a \in R$ (or $a \in Q$) $(a)^\star = (a)$.

Let S be a multiplicatively closed subset of R which strictly contains the set of all regular elements of R. We will define a notion which is slightly stronger than weakly-prime but weaker than semiprime.

Definition 1.8. Let R be a commutative ring and S a multiplicatively subset of R containing all the regular elements of R. Let c be a closure operation defined on $\mathcal{I}(R)$ (or $\mathcal{F}(R)$). We say c is an **S-weakly prime operation** if for all $a \in S$ and all ideals $I \subseteq R$ (or all fractional ideals of R) $aI^\star \subseteq (aI)^\star$. If in addition c is a star operation on $\mathcal{I}(R)$ (or $\mathcal{F}(R)$) we will say that c is an S-prime operation.

Example 1.9. Let $R = k[x]/(x^2)$. Again, because R is Artinian, $Q = R$. By the definition of star operation on either $\mathcal{I}(R)$ or $\mathcal{F}(R)$, all closure operations are star operations on R since for all units u of R and all ideals of R $uI = I$. However, the only principally closed star operation on $\mathcal{I}(R)$ or $\mathcal{F}(R)$ is the identity. The closure c in Example 1.1 is also S-weakly prime for $S = \{0\} \cup U(R)$ where $U(R)$ denotes the units of R.

Example 1.10. Let $R = k[[x, y]]/(x^2, xy)$. The regular elements are again only the units and $uI = I$. Once again, the total ring of fractions is R. So again all closure operations on $\mathcal{I}(R) = \mathcal{F}(R)$ are star operations. It is clear to see the ideals of R are of one of the following forms: (0), (x) or R, (y^n) for some $n \in \mathbb{N}$, (x, y^n) for some $n \in \mathbb{N}$, and $(x+ay^n)$ for some $n \in \mathbb{N}$ and $a \in k^\times$.

However, there are many principally closed star operations on $\mathcal{I}(R)$. They include the identity, $\star$ such that $(x, y^n)^\star = R$ for all natural numbers n and any closure c such that $(x, y^n)^c = (x, y^m)$ for all $n \geq m$ and $(x, y^n)^c = (x, y^n)$ for $n < m$. These non-identity closures are not semiprime.

Those that satisfy $(x, y^n)^c = R$ are also not S-weakly prime for

$$S = \{ax \mid a \in k^\times\} \cup U(R)$$

since $ax(x, y^n) = (0) \not\supseteq (ax) = ax(x, y^n)^c$. However, those that satisfy $(x, y^n)^c \subsetneq (x, y)$ will be S-weakly prime.

2. Counting star and semistar operations

There have been many papers describing domains in terms of the number of star operations or semistar operations defined on them. These include [1,8,9,26,15–18]. For example, if D is a Noetherian domain, D is a one dimensional Gorenstein domain if and only if D has exactly one star operation. This is essentially due to Bass [2] and Matlis [14]. Anderson and Anderson show that if D is a valuation domain, then D has at most two star operations in [1]. Whereas Houston, Mimouni and Park describe all domains which have two or less star operations in [8] and the Noetherian domains which have three star operations [9]. Recently Houston and Park in [10] have determined all Noetherian domains with infinite residue field which have finitely many star operations. They also prove that a Noetherian domain D has a finite number of star operations if and only if the dimension of D is one or less in [9]. White classifies all the star operations on $k+(x^4)k[[x]]$ in [26]. Matsuda and Okabe show that D is a rank one discrete valuation ring if and only if D has precisely two semistar operations. In works of Mimouni and Samman [16–18], the authors find conditions on a domain D which guarantee D has at most five star operations. All these results mentioned are for the domain case where the zero ideal has been excluded. In the domain case, historically the zero ideal was excluded from $\mathcal{F}(R)$ and $\overline{\mathcal{F}}(R)$.

If R is a field, then when we include the zero ideal, we technically have two star operations which are both semistar operations.

- The identity.
- The star operation $(0)^\star = R = R^\star$.

A zero dimensional ring with zero divisors will have a finite number of star (or semistar) operations as long as the number of ideals is finite. In fact, since any star operation on $\mathcal{I}(R)$ is a closure operation it is enough to see the following:

Proposition 2.1. *If R is a commutative ring with n nonzero ideals, then the number of closure operations is at most 2^n.*

Proof. First we prove the following claim.

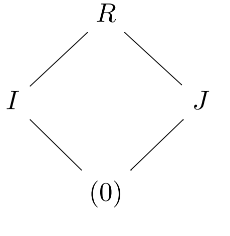
Claim: There are precisely 2^n closure operations if the ideals form a chain. The proof is by induction. If $n = 1$, then the identity and the closure operation sending all ideals to R are the two closure operations. Suppose that $(0) = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = R$ is the chain of ideals in R. Suppose that any ring whose only ideals form a chain of length n has 2^{n-1}

closure operations. Note that the ideals of R/I_1 form a chain $I_1/I_1 \subseteq I_2/I_1 \subseteq \cdots \subseteq I_n/I_1$ of length n so R/I_1 has 2^{n-1} closure operations. By the correspondence theorem, the ideals of R containing I_1 are in one to one correspondence with the ideals of R/I_1. Let $\pi : R \to R/I_1$ be the natural surjection. Suppose c is a closure operation on R/I_1. The only ways to lift c to a closure operation d on R are by defining $I_r^d = f^{-1}((I_r/I_1)^c)$ for $r \geq 1$, and then either setting $I_0^d = I_0$ or $I_0^d = I_1^d$. Thus there are $2^{n-1} \cdot 2 = 2^n$ closure operations on R.

Suppose now that R is a ring with r nonzero ideals which do not form a chain. We will show that R has fewer than 2^r closure operations. Notice, if the ideals do not form a chain, then there are at least one pair of incomparable ideals I and J. There must be a chain of ideals $\mathcal{C}_1 : I \cap J \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n = I + J$ and a chain of ideals $\mathcal{C}_2 : I \cap J \subsetneq J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_m = I + J$ with J_j not comparable to I_i for $j < m$. Note that it is very probable that we have not listed all the ideals. We saw above that there are 2^n closure operations on the set of ideals in $\mathcal{C}_1$. However, note that if we extend any of these closures to the set of all ideals in both $\mathcal{C}_1$ and $\mathcal{C}_2$, then for every closure c among the 2^n closures for which $(I \cap J)^c = I_1^c$, it must be the case that $J_j^c = I + J$. Similarly, there are 2^m closures defined on $\mathcal{C}_2$ and extending any closure d such that $(I \cap J)^d = J_1^d$ to the set of ideals in $\mathcal{C}_1 \cup \mathcal{C}_2$ will force $I_i^d = I + J$. Thus, we have a set of $n+m$ ideals of R and there are only $2^n + 2^m$ closures which is strictly less than 2^{n+m}. Hence, if we extend these closures to the r nonzero ideals on R, we will clearly have less than 2^r closures. $\square$

The following is an illustration of the above proof in the case that there are precisely two maximal ideals which intersect in the zero ideal.

Example 2.2. Suppose the lattice of ideals is



Then the closure operations on R are

(1) $K^c = R$ for all ideals $K \subseteq R$.
(2) $J^c = I^c = R = R^c$ and $(0)^c = (0)$.
(3) $I^c = R = R^c$ and $J^c = J = (0)^c$.
(4) $I^c = R = R^c$ and $J^c = J$ and $(0)^c = (0)$.
(5) $J^c = R = R^c$ and $I^c = I = (0)^c$.
(6) $J^c = R = R^c$ and $I^c = I$ and $(0)^c = (0)$.
(7) $K^c = K$ for all ideals $K \subseteq R$.

Note there are 7 closures, not $8 = 2^3$, where 3 is the number of nonzero ideals.

Note that the above proof actually shows in general that the number of closure operations on a lattice with $n+1$ elements is bounded above by 2^n.

Certainly if R has a finite number of ideals, it isn't surprising that there are a finite number of star or semistar operations. Proposition 1.3 is maybe more surprising. We can also show that $k[[x,y]]/(xy)$ has a finite number of semistar operations of finite type.

3. Semistar operations and standard closures

There is a partial order on the semistar operations given by $\star_1 \leq \star_2$ if $I^{\star_1} \subseteq I^{\star_2}$ for all I. If this is the case $(I^{\star_1})^{\star_2} = I^{\star_2}$ and $(I^{\star_2})^{\star_1} = I^{\star_2}$.

If $\star_1$ and $\star_2$ are semistar operations, we define $\star_1 \cap \star_2$ to be the operation $I^{\star_1 \cap \star_2} = I^{\star_1} \cap I^{\star_2}$. Note that $\star_1 \cap \star_2 \leq \star_1, \star_2$.

Fact 3.1. If $\star_1$, $\star_2$ are star (respectively semistar) operations then $\star_1 \cap \star_2$ is a star (respectively semistar) operation.

Proof. First we will show $\star_1 \cap \star_2$ is a closure operation. For extension, since $I \subseteq I^{\star_1}$ and $I \subseteq I^{\star_2}$ for all ideals I, $I \subseteq I^{\star_1} \cap I^{\star_2}$ for all ideals I. For order preservation, we know that if $I \subseteq J$, $I^{\star_1} \cap I^{\star_2} \subseteq J^{\star_1} \cap J^{\star_2}$. For idempotence, since $\star_1 \cap \star_2 \leq \star_1$ and $\star_1 \cap \star_2 \leq \star_2$ we have $(I^{\star_1 \cap \star_2})^{\star_1} \cap (I^{\star_1 \cap \star_2})^{\star_2} = I^{\star_1} \cap I^{\star_2}$. For all regular elements a, $(aI)^{\star_1} \cap (aI)^{\star_2} = aI^{\star_1} \cap aI^{\star_2} = a(I^{\star_1} \cap I^{\star_2})$. Where the first equality holds since $\star_1$ and $\star_2$ are divisible closure operations and the second inequality holds since a is a regular element. Thus $\star_1 \cap \star_2$ is a divisible closure operation. If $\star_i$ are star operations for $i = 1, 2$ then $R^{\star_i} = R$. So $R \subseteq R^{\star_1 \cap \star_2} \subseteq R^{\star_1} = R$ which implies $R^{\star_1 \cap \star_2} = R$. $\square$

We will use Epstein's [4] standard closures to see that $k[[x,y]]/(xy)$ has finitely many semistar operations of finite type.

Definition 3.2. A closure $c : \mathcal{I}(R) \to \mathcal{I}(R)$ is **standard** if $((xI)^c : x) = I^c$ for any regular element $x \in R$.

Definition 3.3. A closure operation $c : \mathcal{A} \to \mathcal{A}$ is of **finite-type** if

$$A^c = \bigcup \{I^c \mid \text{for all finitely generated } I \in \mathcal{A} \text{ with } I \subseteq A\}.$$

Theorem 3.4. *(See [4].) There is a one to one order preserving correspondence between the set of finite-type standard closure operations on R and the finite-type semistar operations on R.*

Counting the number of finite-type semistar operations on a ring R amounts to counting the finite type standard closures defined on R. Our main result is to classify the standard operations on the ring $R = k[[x,y]]/(xy)$.

In the proofs we will frequently use the following fact pertaining to all closure operations:

Fact 3.5. Let R be a commutative ring and c a closure operation defined on the ideals of R. If $J \subseteq I \subseteq J^c$, then $I^c = J^c$.

Before stating the classification theorem, we state and prove several computational Lemmas which will be exploited in the proof of the theorem. The first such lemma allows us to conclude that a standard closure $\star$ fixes an (x^n) (or (y^n)) for some natural number $n > 1$, then $\star$ fixes all ideals of the form (x^n) (or (y^n)).

Lemma 3.6. *Let $R = k[[x, y]]/(xy)$. If $\star$ is a standard closure and $(x^n)^\star = (x^n)$ (respectively $(y^n)^\star = (y^n)$) for some $n \in \mathbb{N}$ with $n > 1$ then $(x^r)^\star = (x^r)$ (respectively $(y^r)^\star = (y^r)$) for all $r \in \mathbb{N}$.*

Proof. Suppose $(x^n)^\star = (x^n)$ for some $n \in \mathbb{N}$. Consider

$$(((x + y)(x^{n-1}))^\star : x + y) = ((x^n)^\star : x + y) = ((x^n) : x + y) = (x^{n-1}).$$

Since $\star$ is standard $(x^{n-1})^\star = (x^{n-1})$. By induction we can see $(x^r)^\star = (x^r)$ for all $r \leq n$. By the extension property, $(x^{n+1})^\star \subseteq (x^n)^\star = (x^n)$. Now suppose $(x^{n+1})^\star = (x^n)$. We have

$$(((x + y)(x^n))^\star : x + y) = ((x^{n+1})^\star : x + y) = ((x^n) : x + y) = (x^{n-1}).$$

Since $\star$ is standard this indicates $(x^n)^\star = (x^{n-1})$ which is a contradiction. By induction we can see $(x^r)^\star = (x^r)$ for all $r > n$. $\square$

The following two lemmas allow us to conclude that if $\star$ is a standard closure and there is some natural number such that $(x^n)^\star \neq (x^n)$ (or $(y^n)^\star \neq (y^n)$), then for all natural numbers $n > 1$ $(x^n)^\star \neq (x^n)$ (or $(y^n)^\star \neq (y^n)$). The second Lemma also allows us to make some conclusions about the (x, y)-primary ideals.

Lemma 3.7. *Let $R = k[[x, y]]/(xy)$. If $\star$ is a standard closure and $(x^n)^\star = (x^i)$ (respectively $(y^n)^\star = (y^i)$) for some $n, i \in \mathbb{N}, i < n$ then $(x^r)^\star = (x)$ (respectively $(y^r)^\star = (y)$) for all $r \in \mathbb{N}$.*

Proof. Suppose $(x^n)^\star = (x^i)$ for some $i < n$. For $i \leq r \leq n$,

$$(x^i) = (x^n)^\star \subseteq (x^r)^\star \subseteq (x^i)^\star = (x^i)$$

implies that $(x^r)^\star = (x^i)$. Consider

$$(((x + y)(x^i))^\star : x + y) = ((x^{i+1})^\star : x + y) = ((x^i) : x + y) = \begin{cases} (x) \text{ if } i = 1 \\ (x^{i-1}) \text{ if } i > 1. \end{cases}$$

Clearly, if $i > 1$, then $\star$ is not standard. Thus $i = 1$. Clearly $(x^r)^\star = (x)$ for all $1 \leq r \leq n$. We need to see that $(x^r)^\star = (x)$ for all $r \geq 1$. We will show this using induction. Suppose $(x^r)^\star = (x)$. We need only see that $(x^{r+1})^\star = (x)$. Since standard closures are weakly prime

$$(x^2) = (x+y)(x^r)^\star \subseteq ((x+y)(x^r))^\star = (x^{r+1})^\star.$$

Applying $\star$ to the sequence of containments we obtain $(x) = (x^2)^\star \subseteq (x^{r+1})^\star$. So $(x) \subseteq (x^{r+1})^\star$. So either $(x^{r+1})^\star = (x)$, $(x^{r+1})^\star = (x, y^j), j \in \mathbb{N}$, or $(x^{r+1})^\star = R$. If $(x^{r+1})^\star = (x, y^j)$ then

$$(((x+y)(x^r))^\star : x+y) = ((x^{r+1})^\star : x+y) = ((x, y^j) : x+y) = \begin{cases} (x, y^{j-1}) \text{ if } j > 1 \\ (x, y) \text{ if } j = 1. \end{cases}$$

If $(x^{r+1})^\star = R$ then

$$(((x+y)(x^r))^\star : x+y) = ((x^{r+1})^\star : x+y) = (R : x+y) = R.$$

Since both of these contradict standardness, it must be the case $(x^{r+1})^\star = (x)$. So $(x^r)^\star = (x)$ for all $r \geq 1$. □

Lemma 3.8. *Let $R = k[[x, y]]/(xy)$. If $\star$ is a standard closure and $(x^n)^\star = R$ (respectively $(y^n)^\star = R$) for some $n \in \mathbb{N}$ then $(x^r)^\star = (x^r + ay^s)^\star = (x^r, y^s)^\star = R$ (respectively $(y^s)^\star = (x^r + ay^s)^\star = (x^r, y^s)^\star = R$) for all $r, s \in \mathbb{N}$.*

Proof. Suppose $(x^n)^\star = R$ for some $n \in \mathbb{N}$. Since $R = (x^n)^\star \subseteq (x^t)^\star \subseteq R$ for all $t \leq n$ we have $(x^t)^\star = R$ for all $t \leq n$. Since $\star$ is standard, it is weakly prime. Thus

$$(x+y)(x^n)^\star \subseteq ((x+y)(x^n))^\star \Rightarrow (x+y) \subseteq (x^{n+1})^\star.$$

Hence $(x^{n+1})^\star = (x+y)^\star$.

Note if $n \geq 2$,

$$R = (x^n)^\star \subseteq (x+y)^\star = (x^{n+1})^\star \subseteq R.$$

So $(x^{n+1})^\star = R$. By induction $(x^r)^\star = R$ for all $r \in \mathbb{N}$.

Suppose $n = 1$. We have $(x)^\star = R$. Either $(x^2)^\star = (x^2)$, $(x^2)^\star = (x^2, y^j), j \in \mathbb{N}$, $(x^2)^\star = (x, y^j)$, $(x^2)^\star = (x + ay), a \in k^\times$ or $(x^2)^\star = R$.

Suppose $(x^2)^\star = (x^2)$. We have

$$(((x+y)(x))^\star : x+y) = ((x^2)^\star : x+y) = ((x^2) : x+y) = (x).$$

Since $\star$ is standard this implies $(x)^\star = (x)$, which is a contradiction.

Suppose $(x^2)^\star = (x^2, y^j)$. We have

$$(((x+y)(x))^\star : x+y) = ((x^2)^\star : x+y) = ((x^2, y^j) : x+y) = \begin{cases} (x, y^{j-1}) \text{ if } j > 1 \\ (x, y) \text{ if } j = 1. \end{cases}$$

Since $\star$ is standard both cases give a contradiction.

Suppose $(x^2)^\star = (x, y^j)$. Now $R = (x)^\star \subseteq (x, y^j)^\star \subseteq R$ which implies $(x, y^j)^\star = R$. This is a contradiction because (x, y^j) must be $\star$ closed.

Suppose $(x^2)^\star = (x + ay^j), a \in k^\times$. We have

$$(((x+y)(x))^\star : x+y) = ((x^2)^\star : x+y) = ((x+ay^j) : x+y) = \begin{cases} (x, y^j) \text{ if } j > 1 \\ R \text{ if } j = 1, a = 1 \\ (x, y) \text{ if } j = 1, a \neq 1. \end{cases}$$

Since $\star$ is standard the first and third cases obviously give a contradiction. If $(x)^\star = R$ then $j = 1, a = 1$ so $(x^2)^\star = (x + y)$. Since $\star$ is weakly prime we have

$$(x + y^2)(x)^\star \subseteq ((x + y^2)(x))^\star \Rightarrow (x + y^2) \subseteq (x^2)^\star.$$

And then $(x + y^2) \subseteq (x^2)^\star = (x + y)$ which is a contradiction. Thus $(x^2)^\star = R$. And by above $(x^r)^\star = R$ for all $r \in \mathbb{N}$.

Since $R = (x^{r+1})^\star \subseteq (x^r + ay^s)^\star \subseteq (x^r, y^s)^\star \subseteq R$ we have $(x^r + ay^s)^\star = (x^r, y^s)^\star = R$ for all $r, s \in \mathbb{N}$. $\square$

Similarly to the last two lemmas, if $\star$ is a standard closure and we know that the maximal ideal is not closed then no (x, y)-primary ideals will be closed. We thank the referee for suggesting the clearer proof below.

Lemma 3.9. *Let $R = k[[x, y]]/(xy)$. If $\star$ is a standard closure and $(x, y)^\star = R$ then*

$$(x^m, y^n)^\star = (x^m + ay^n)^\star = R$$

for all $m, n \in \mathbb{N}$ and $a \in k^\times$.

Proof. Since $\star$ is standard, it is weakly prime. Thus we have

$$(x^{m+1}, y^{n+1}) \subseteq (x^m + ay^n) = (x^m + ay^n)(x, y)^\star \subseteq (x^{m+1}, y^{n+1})^\star,$$

for any m, $n \in \mathbb{N}$ and $a \in k^\times$. Applying $\star$ to the above shows that

$$(x^{m+1}, y^{n+1})^\star = (x^m + ay^n)^\star.$$

Next note that

$$(x, y) = (x + y^2, x^2 + y) \subseteq (x^2, y^3)^\star + (x^3, y^2)^\star \subseteq (x^2, y^2)^\star.$$

Since $(x, y)^\star = R$, it follows that $(x^2, y^2)^\star = R$. Moreover, $(x + y)^\star = (x^2, y^2)^\star = R$.

We next show by induction that for all n, $(x^n, y^n)^\star = R$. So suppose this is true for some n. Then since $\star$ is weakly prime,

$$(x+y) = (x+y)(x^n, y^n)^\star \subseteq (x^{n+1}, y^{n+1})^\star.$$

But since $(x+y)^\star = R$, it follows that $(x^{n+1}, y^{n+1})^\star = R$, completing the induction.

Now for any $m > n$, $(x^m, y^n)^\star \supseteq (x^m, y^m)^\star = R$. If $m < n$, then

$$(x^m, y^n)^\star \supseteq (x^n, y^n)^\star = R.$$

So for all $m, n \in \mathbb{N}$, we have $(x^n, y^n)^\star = R$.

Finally, for any $m, n \in \mathbb{N}$ and $a \in k^\times$, $(x^m + ay^n)^\star = (x^{m+1}, y^{n+1})^\star = R$. $\square$

The next lemma allows us to determine the possibilities for the closures of the monomial (x, y)-primary ideals if the maximal ideal is closed for a standard closure $\star$.

Lemma 3.10. *Let $R = k[[x, y]]/(xy)$. If $\star$ is a standard closure operation and $(x, y)^\star = (x, y)$ then either*

(i) $(x^m, y^n)^\star = (x^m, y^n)$ *for all* $m, n \in \mathbb{N}$,
(ii) $(x^m, y^n)^\star = (x, y^n)$ *for all* $m, n \in \mathbb{N}$ *or*
(iii) $(x^m, y^n)^\star = (x^m, y)$ *for all* $m, n \in \mathbb{N}$.

Proof. Since $(x^2, y^2)^\star \subseteq (x, y)^\star = (x, y)$ then either (1) $(x^2, y^2)^\star = (x^2, y^2)$, (2) $(x^2, y^2)^\star = (x, y^2)$, (3) $(x^2, y^2)^\star = (x^2, y)$, (4) $(x^2, y^2)^\star = (x + ay), a \in k^\times$, or (5) $(x^2, y^2)^\star = (x, y)$.

(4) Suppose $(x^2, y^2)^\star = (x + ay), a \in k^\times$. Then

$$(((x + ay)(x, y))^\star : x + ay) = ((x^2, y^2)^\star : x + ay) = ((x + ay) : x + ay) = R.$$

Since $\star$ is standard this implies $(x, y)^\star = R$ which is a contradiction.

(5) Suppose $(x^2, y^2)^\star = (x, y)$. Then

$$(((x + y)(x, y))^\star : x + y) = ((x^2, y^2)^\star : x + y) = ((x, y) : x + y) = R.$$

So again we have a contradiction.

Thus only possibilities (1)–(3) remain.

(1) Suppose $(x^2, y^2)^\star = (x^2, y^2)$. Since $(x^3, y^3)^\star \subseteq (x^2, y^2)^\star = (x^2, y^2)$, we have five possibilities: (a) $(x^3, y^3)^\star = (x^2, y^3)$, (b) $(x^3, y^3)^\star = (x^3, y^2)$, (c) $(x^3, y^3)^\star = (x^2, y^2)$, (d) $(x^3, y^3)^\star = (x^2 + ay^2)$, for $a \in k^\times$ or (e) $(x^3, y^3)^\star = (x^3, y^3)$.

(a) Suppose $(x^3, y^3)^\star = (x^2, y^3)$. Then

$$(((x + y)(x^2, y^2))^\star : x + y) = ((x^3, y^3)^\star : x + y) = ((x^2, y^3) : x + y) = (x, y^2).$$

Since $\star$ is standard this implies $(x^2, y^2)^\star = (x, y^2)$ which is a contradiction.

(b) Suppose $(x^3, y^3)^\star = (x^3, y^2)$. Then

$$(((x+y)(x^2, y^2))^\star : x+y) = ((x^3, y^3)^\star : x+y) = ((x^3, y^2) : x+y) = (x^2, y)$$

which again leads to a contradiction.

(c) Suppose $(x^3, y^3)^\star = (x^2, y^2)$. Then

$$(((x+y)(x^2, y^2))^\star : x+y) = ((x^3, y^3)^\star : x+y) = ((x^2, y^2) : x+y) = (x, y)$$

which is a contradiction.

(d) Suppose $(x^3, y^3)^\star = (x^2 + ay^2), a \in k^\times$. Then

$$(((x+y)(x^2, y^2))^\star : x+y) = ((x^3, y^3)^\star : x+y) = ((x^2 + ay^2) : x+y) = (x + ay)$$

also a contradiction.

Thus we are left with (e) $(x^3, y^3)^\star = (x^3, y^3)$. By induction we see for all $n \in \mathbb{N}$ that $(x^n, y^n)^\star = (x^n, y^n)$.

Since $(x, y^2)^\star \subseteq (x, y)^\star = (x, y)$ we have either $(x, y^2)^\star = (x, y^2)$ or $(x, y^2)^\star = (x, y)$. Suppose $(x, y^2)^\star = (x, y)$. Since $\star$ is weakly prime we have

$$(x+y)(x, y^2)^\star \subseteq ((x+y)(x, y^2))^\star$$

implies $(x^2, y^2) \subseteq (x^2, y^3)^\star$. Thus $(x^2, y^3)^\star = (x^2, y^2)^\star = (x^2, y^2)$. Now

$$(((x^3 + y^2)(x^2, y^3))^\star : x^3 + y^2) = ((x^5, y^5)^\star : x^3 + y^2) = ((x^5, y^5) : x^3 + y^2) = (x^2, y^3).$$

Since $\star$ is standard this implies $(x^2, y^3)^\star = (x^2, y^3)$ which is a contradiction. Thus $(x, y^2)^\star = (x, y^2)$ and we can show similarly that $(x^2, y)^\star = (x^2, y)$.

Suppose $(x, y^k)^\star = (x, y^k)$ for all $k < n$, for some $n \geq 2$. Since

$$(x, y^n)^\star \subseteq (x, y^{n-1})^\star = (x, y^{n-1})$$

we have either $(x, y^n)^\star = (x, y^n)$ or $(x, y^n)^\star = (x, y^{n-1})$. Suppose $(x, y^n)^\star = (x, y^{n-1})$. Since $\star$ is weakly prime we have $(x^{n-1} + y)(x, y^n)^\star \subseteq ((x^{n-1} + y)(x, y^n))^\star$ implies $(x^n, y^n) \subseteq (x^n, y^{n+1})^\star$. Thus $(x^n, y^{n+1})^\star = (x^n, y^n)^\star = (x^n, y^n)$. Now

$$\begin{aligned}(((x^{n+1} + y^n)(x^n, y^{n+1}))^\star : x^{n+1} + y^n) &= ((x^{2n+1}, y^{2n+1})^\star : x^{n+1} + y^n) \\ &= ((x^{2n+1}, y^{2n+1}) : x^{n+1} + y^n) = (x^n, y^{n+1}).\end{aligned}$$

Since $\star$ is standard this implies $(x^n, y^{n+1})^\star = (x^n, y^{n+1})$ which is a contradiction. Thus for all $n \in \mathbb{N}$ $(x, y^n)^\star = (x, y^n)$ and we can similarly see that $(x^n, y)^\star = (x^n, y)$ for all $n \in \mathbb{N}$. So $(x^m, y^n)^\star \subseteq (x, y^n)^\star = (x, y^n)$ and $(x^m, y^n)^\star \subseteq (x^m, y)^\star = (x^m, y)$. Thus $(x^m, y^n) \subseteq (x^m, y^n)^\star \subseteq (x, y^n) \cap (x^m, y) = (x^m, y^n)$. So $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$.

(2) Suppose $(x^2, y^2)^\star = (x, y^2)$. This is the base case. Now suppose $(x^k, y^2)^\star = (x, y^2)$ for all $k < n$, for some $n \geq 2$. Since $\star$ is weakly prime we have

$$(x+y)(x^{n-1}, y^2)^\star \subseteq ((x+y)(x^{n-1}, y^2))^\star$$

which implies $(x^2, y^3) \subseteq (x^n, y^3)^\star$. Thus

$$(x^2, y^3) \subseteq (x^n, y^3)^\star \subseteq (x^n, y^2)^\star \subseteq (x^{n-1}, y^2)^\star = (x, y^2).$$

Since $(x^2, y^3) + (x^n, y^2) = (x^2, y^2)$ we have

$$(x, y^2) = (x^2, y^2)^\star \subseteq (x^n, y^2)^\star \subseteq (x, y^2).$$

This implies $(x^n, y^2)^\star = (x, y^2)$ for all $n \in \mathbb{N}$. Also

$$(((x+y)(x^n, y))^\star : x+y) = ((x^{n+1}, y^2)^\star : x+y) = ((x, y^2) : x+y) = (x, y).$$

Since $\star$ is standard $(x^n, y)^\star = (x, y)$ for all $n \in \mathbb{N}$.

Now suppose $(x^m, y^k)^\star = (x, y^k)$ for all $m \in \mathbb{N}$ and $2 \leq k < n$. First we will consider those cases where $m \geq 2$. Since $(x^m, y^n)^\star \subseteq (x^m, y^{n-1})^\star = (x, y^{n-1})$ we have either $(x^m, y^n)^\star = (x, y^{n-1})$ or $(x^m, y^n)^\star = (x^j, y^n)$ for $1 \leq j \leq m$.

Suppose $(x^m, y^n)^\star = (x, y^{n-1})$. Since $\star$ is weakly prime we have

$$(x+y)(x^m, y^n)^\star \subseteq ((x+y)(x^m, y^n))^\star$$

implies $(x^2, y^n) \subseteq (x^{m+1}, y^{n+1})^\star$. Hence $(x^{m+1}, y^{n+1})^\star = (x^2, y^n)^\star$. Since

$$(x^{m+1}, y^{n+1})^\star \subseteq (x^m, y^n)^\star \subseteq (x^2, y^n)^\star$$

we have $(x^{m+1}, y^{n+1})^\star = (x^m, y^n)^\star = (x, y^{n-1})$. Now

$$(((x+y)(x^m, y^n))^\star : x+y) = ((x^{m+1}, y^{n+1})^\star : x+y) = ((x, y^{n-1}) : x+y) = (x, y^{n-2}).$$

Since $\star$ is standard this implies $(x^m, y^n)^\star = (x, y^{n-2})$ which is a contradiction.

Suppose $(x^m, y^n)^\star = (x^j, y^n)$ for $1 \leq j \leq m$. Consider the case where $j > 1$. We have

$$(((x+y)(x^{j-1}, y^{n-1}))^\star : x+y) = ((x^j, y^n)^\star : x+y) = ((x^j, y^n) : x+y) = (x^{j-1}, y^{n-1}).$$

This implies $(x^{j-1}, y^{n-1})^\star = (x^{j-1}, y^{n-1})$. By induction $(x^{j-1}, y^{n-1})^\star = (x, y^{n-1})$ implying $j = 2$. So we have $(x^m, y^n)^\star = (x^2, y^n)$.

Since $(x^{m+1}, y^{n+1})^\star \subseteq (x^m, y^n)^\star = (x^2, y^n)$ either we have (a) $(x^{m+1}, y^{n+1})^\star = (x^{m+1}, y^{n+1})$, (b) $(x^{m+1}, y^{n+1})^\star = (x^{m+1}, y^n)$, (c) $(x^{m+1}, y^{n+1})^\star = (x^m, y^{n+1})$, (d) $(x^{m+1}, y^{n+1})^\star = (x^2, y^n)$ or (e) $(x^{m+1}, y^{n+1})^\star = (x^m + ay^n), a \in k^\times$.

(a) Suppose $(x^{m+1}, y^{n+1})^\star = (x^{m+1}, y^{n+1})$. Then

$$(((x+y)(x^m,y^n))^\star : x+y) = ((x^{m+1},y^{n+1})^\star : x+y)$$
$$= ((x^{m+1},y^{n+1}) : x+y) = (x^m,y^n).$$

Since $\star$ is standard, $(x^2,y^n) = (x^m,y^n)^\star = (x^m,y^n)$ implying $m=2$. We conclude that

$$(x^3,y^{n+1})^\star = (x^3,y^{n+1})$$

and have

$$(((x+y^2)(x^2,y^{n-1}))^\star : x+y^2) = ((x^3,y^{n+1})^\star : x+y^2) = ((x^3,y^{n+1}) : x+y^2)$$
$$= (x^2,y^{n-1}).$$

Since $\star$ is standard this implies $(x^2,y^{n-1}))^\star = (x^2,y^{n-1})$. This contradicts the inductive hypothesis $(x^2,y^{n-1}))^\star = (x,y^{n-1})$.

(b) Suppose $(x^{m+1},y^{n+1})^\star = (x^{m+1},y^n)$. Then

$$(((x+y)(x^m,y^n))^\star : x+y) = ((x^{m+1},y^{n+1})^\star : x+y) = ((x^{m+1},y^n) : x+y)$$
$$= (x^m,y^{n-1}).$$

Since $\star$ is standard this implies $(x^m,y^n))^\star = (x^m,y^{n-1})$ which is a contradiction.

(c) Suppose $(x^{m+1},y^{n+1})^\star = (x^m,y^{n+1})$. Then

$$(((x+y)(x^m,y^n))^\star : x+y) = ((x^{m+1},y^{n+1})^\star : x+y) = ((x^m,y^{n+1}) : x+y)$$
$$= (x^{m-1},y^n).$$

Since $\star$ is standard, $(x^2,y^n) = (x^m,y^n)^\star = (x^{m-1},y^n)$. This implies $m=3$. So $(x^3,y^n)^\star = (x^2,y^n)$ and $(x^4,y^{n+1})^\star = (x^3,y^{n+1})$. We have

$$(((x^2+y)(x^2,y^n))^\star : x^2+y) = ((x^4,y^{n+1})^\star : x^2+y) = ((x^3,y^{n+1}) : x^2+y) = (x,y^n).$$

Since $\star$ is standard this implies $(x^2,y^n)^\star = (x,y^n)$. This is a contradiction since (x^2,y^n) must be $\star$-closed.

(d) Suppose $(x^{m+1},y^{n+1})^\star = (x^2,y^n)$. Then

$$(((x+y)(x^m,y^n))^\star : x+y) = ((x^{m+1},y^{n+1})^\star : x+y) = ((x^2,y^n) : x+y) = (x,y^{n-1}).$$

Since $\star$ is standard this implies $(x^m,y^n)^\star = (x,y^{n-1})$ which is a contradiction.

(e) Suppose $(x^{m+1},y^{n+1})^\star = (x^m+ay^n), a \in k^\times$. Then

$$(((x+y)(x^m,y^n))^\star : x+y) = ((x^{m+1},y^{n+1})^\star : x+y) = ((x^m+ay^n) = (x^{m-1}+ay^{n-1}).$$

Since $\star$ is standard this implies $(x^m, y^n))^\star = (x^{m-1} + ay^{n-1})$ which is a contradiction. Thus $j = 1$ and $(x^m, y^n)^\star = (x, y^n)$ for all $m \geq 2$ and $n \in \mathbb{N}$. For the case where $m = 1$ consider

$$(((x + y)(x, y^n))^\star : x + y) = ((x^2, y^{n+1})^\star : x + y) = ((x, y^{n+1}) : x + y) = (x, y^n).$$

Since $\star$ is standard this implies $(x, y^n)^\star = (x, y^n)$. Hence $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$.

(3) If we suppose $(x^2, y^2)^\star = (x^2, y)$ then by exchanging the roles of x and y and repeating the proof of (2) we obtain $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$. $\square$

If the monomial (x, y)-primary ideals are not closed, the following Lemma allows us to determine the closures of the principal (x, y)-primary ideals.

Lemma 3.11. *Let $R = k[[x, y]]/(xy)$. If $\star$ is a standard closure operation and $(x^m, y^n)^\star = (x, y^n)$ (respectively $(x^m, y^n)^\star = (x^m, y)$) for all $m, n \in \mathbb{N}$ then $(x^m + ay^n)^\star = (x, y^n)$ (respectively $(x^m + ay^n)^\star = (x^m, y)$) for all $m, n \in \mathbb{N}$ and $a \in k^\times$.*

Proof. We have

$$(x^{m+1}, y^{n+1}) \subseteq (x^m + ay^n) \subseteq (x^m, y^n).$$

Applying $\star$ to the chain we obtain

$$(x, y^{n+1}) \subseteq (x^m + ay^n)^\star \subseteq (x, y^n).$$

Since $(x, y^{n+1}) + (x^m + ay^n) \subseteq (x^m + ay^n)^\star$ and $(x, y^{n+1}) + (x^m + ay^n) = (x, y^n)$,

$$(x^m + ay^n)^\star = (x, y^n). \qquad \square$$

Lemma 3.12. *Let $R = k[[x, y]]/(xy)$. If $\star$ is a standard closure operation and $(y^s)^\star = (x^i, y^j)$ (respectively $(x^s)^\star = (x^i, y^j)$) for some natural numbers s, i, j, then $(y^t)^\star = (x, y^t)$ (respectively $(x^t)^\star = (x^t, y)$) for all natural numbers t.*

Proof. Now suppose that $(y^s)^\star = (x^j, y^k)$ for some $s, i, j \in \mathbb{N}$. So $(x^i, y^j) = (y^s)^\star \subseteq (y^j)^\star$ implies $(y^j)^\star = (x^i, y^j)^\star = (x^i, y^j)$. Suppose $j < s$. We have

$$\begin{aligned}(((x + y^{s-j})(y^j)j^\star : x + y^{s-k}) &= ((y^s)^\star : x + y^{s-j}) \\ &= ((x^i, y^j) : x + y^{s-j}) = \begin{cases} (x^{i-1}, y) \text{ if } s > 2j, i > 1 \\ (x^{i-1}, y^{2j-s}) \text{ if } s < 2j, i > 1 \\ R \text{ if } s \geq 2j, i = 1 \\ (x, y^{2j-s}) \text{ if } s < 2j, i = 1 \end{cases}.\end{aligned}$$

The first three cases are obvious contradictions of $\star$ being standard. For the last case, since it must be the case that $(x, y^{2j-s}) = (x^i, y^j)$ we see that $2j - s = j$ which in turn implies $s = j$. Thus $(y^s)^\star = (x, y^s)$. Since

$$(x, y^s) = (y^s)^\star \subseteq (x^m, y^s)^\star \subseteq (x, y^s)^\star = (x, y^s),$$

then for all natural numbers m, $(x^m, y^s)^\star = (x, y^s)^\star$. We can now conclude from Lemma 3.9 that $(x, y)^\star \neq R$, since there is a pair $(m, s) \in \mathbb{N}^2$ with $(x^m, y^s)^\star \neq R$. Thus $(x, y)^\star = (x, y)$. Now applying Lemma 3.10 we see that $(x^m, y^n)^\star = (x, y^n)$ for all natural numbers m and n since the other two cases would lead us to a contradiction. Now using this fact we will show that $(y^t)^\star = (x, y^t)$ for all natural numbers t.

Suppose first that $t < s$. Then

$$(((x + y^t)(y^{s-t}))^\star : x + y^t) = ((y^s)^\star : x + y^t) = ((x, y^s) : x + y^t) = (x, y^{s-t}).$$

Since $\star$ is standard $(y^{s-t}))^\star = (x, y^{s-t})$. To show $(y^t)^\star = (x, y^t)$ for all t it is enough to show $(y^{s+1})^\star = (x, y^{s+1})$. Since $(y^{s+1})^\star \subseteq (y^s)^\star = (x, y^s)$ either $(y^{s+1})^\star = (x, y^s)$ or $(y^{s+1})^\star = (x^r, y^{s+1})$ for some $r \geq 1$. Suppose $(y^{s+1})^\star = (x, y^s)$. We have

$$(((x + y)(y^s))^\star : x + y) = ((y^{s+1})^\star : x + y) = ((x, y^s) : x + y) = (x, y^{s-1}).$$

This implies $(y^s)^\star = (x, y^{s-1})$ which is a contradiction. Suppose $(y^{s+1})^\star = (x^r, y^{s+1})$ for some $r > 1$. We have

$$(((x + y)(y^s))^\star : x + y) = ((y^{s+1})^\star : x + y) = ((x^r, y^{s+1}) : x + y) = (x^{r-1}, y^s).$$

Since $\star$ is standard $(y^s)^\star = (x^{r-1}, y^s)$ which implies $r = 2$. So $(y^{s+1})^\star = (x^2, y^{s+1})$. Lemma 3.10 implies that the ideals (x^m, y^n) are $\star$-closed for all m and n $(x^2, y^{s+1})^\star = (x, y^{s+1})$ which is a contradiction since (x^2, y^{s+1}) must be $\star$ closed. So $r = 1$ and $(y^{s+1})^\star = (x, y^{s+1})$. Thus $(y^t)^\star = (x, y^t)$ for all t.

To show that if $(x^s)^\star = (x^i, y^j)$ for some natural numbers i, j and s then $(x^t)^\star = (x^t, y)$ for all natural numbers t, we need only reverse the roles of x and y in the proof. $\square$

Lemma 3.13. *Let $R = k[[x, y]]/(xy)$. If $\star$ is a standard closure operation, it cannot be the case that $(x^n)^\star = (x^i + ay^j)$ or $(y^n)^\star = (x^i + ay^j)$ for any natural numbers i, j and n.*

Proof. Suppose that $(x^m)^\star = (x^i + ay^j)$ for some natural numbers i, j and m and some $a \in k^\times$. This cannot be a standard closure. If $(x, y)^\star = R$ then by Lemma 3.9 $(x^i + ay^j)^\star = R$ which is a contradiction since $(x^i + ay^j)$ must be $\star$-closed. If $(x, y)^\star = (x, y)$ then by 3.10 there are three possibilities. If we suppose $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$ by Lemma 3.11 then $(x^i + ay^j)$ is not $\star$-closed. So suppose $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$. Now

$$(x^i + ay^j) = (x^m)^\star \subseteq (x^m, y^n)^\star = (x^m, y^n).$$

This implies $i \geq m$ hence $(x^m) \nsubseteq (x^i + ay^j)$ which is a contradiction of the extension property. □

Now we are ready to state and prove the Main Classification Theorem:

Theorem 3.14. *Let $R = k[[x, y]]/(xy)$. There are 24 standard closures on R.*

(1) $\star_1$: *The identity.*
(2) $\star_2$: *$I^{\star_2} = I$ for all $I \neq (x^n + ay^m)$ and $(x^n + ay^m)^{\star_2} = (x^n, y^m)$ for natural numbers n and m and $a \in k^\times$.*
(3) $\star_3$: *$(0)^{\star_3} = (0)$; $(x^n)^{\star_3} = (x^n)$ and $(y^n)^{\star_3} = (y^n)$ for all natural numbers n;*

$$(x^n, y^m)^{\star_3} = (x^n + ay^m)^{\star_3} = (x, y^m)$$

for all natural numbers m and n, and all $a \in k^\times$; and $R^{\star_3} = R$.
(4) $\star_4$: *$(0)^{\star_4} = (0)$; $(x^n)^{\star_4} = (x)$ and $(y^n)^{\star_4} = (y^n)$ for all natural numbers n;*

$$(x^n, y^m)^{\star_4} = (x^n + ay^m)^{\star_4} = (x, y^m)$$

for all natural numbers m and n, and all $a \in k^\times$; and $R^{\star_4} = R$.
(5) $\star_5$: *$(0)^{\star_5} = (0)$; $(x^n)^{\star_5} = (x^n)$ for all natural numbers n;*

$$(y^m)^{\star_5} = (x^n, y^m)^{\star_5} = (x^n + ay^m)^{\star_5} = (x, y^m)$$

for all natural numbers m and n and all $a \in k^\times$; and $R^{\star_5} = R$.
(6) $\star_6$: *$(0)^{\star_6} = (0)$; $(x^n)^{\star_6} = (x)$ for all natural numbers n;*

$$(y^m)^{\star_6} = (x^n, y^m)^{\star_6} = (x^n + ay^m)^{\star_6} = (x, y^m)$$

for all natural numbers m and n and all $a \in k^\times$; and $R^{\star_6} = R$.
(7) $\star_7$: *$(0)^{\star_7} = (x^n)^{\star_7} = (x)$ for all natural numbers n;*

$$(y^m)^{\star_7} = (x^n, y^m)^{\star_7} = (x^n + ay^m)^{\star_7} = (x, y^m)$$

for all natural numbers m and n and all $a \in k^\times$; and $R^{\star_7} = R$.
(8) $\star_8$: *$(0)^{\star_8} = (0)$; $(y^n)^{\star_8} = (y^n)$ and $(x^n)^{\star_8} = (x^n)$ for all natural numbers n;*

$$(x^m, y^n)^{\star_8} = (x^m + ay^n)^{\star_8} = (x^m, y)$$

for all natural numbers m and n and all $a \in k^\times$; $R^{\star_8} = R$.
(9) $\star_9$: *$(0)^{\star_9} = (0)$; $(y^n)^{\star_9} = (y)$ and $(x^n)^{\star_9} = (x^n)$ for all natural numbers n;*

$$(x^m, y^n)^{\star_9} = (x^m + ay^n)^{\star_9} = (x^m, y)$$

for all natural numbers m and n and all $a \in k^\times$; $R^{\star_9} = R$.

(10) $\star_{10}$: $(0)^{\star_{10}} = (0)$; $(y^n)^{\star_{11}} = (y^n)$ *for all natural numbers* n;

$$(x^m)^{\star_{10}} = (x^m, y^n)^{\star_{10}} = (x^m + ay^n)^{\star_{10}} = (x^m, y)$$

for all natural numbers m *and* n *and all* $a \in k^\times$; $R^{\star_{10}} = R$.

(11) $\star_{11}$: $(0)^{\star_{11}} = (0)$; $(y^n)^{\star_{11}} = (y)$ *for all natural numbers* n;

$$(x^m)^{\star_{11}} = (x^m, y^n)^{\star_{11}} = (x^m + ay^n)^{\star_{11}} = (x^m, y)$$

for all natural numbers m *and* n *and all* $a \in k^\times$; $R^{\star_{11}} = R$.

(12) $\star_{12}$: $(0)^{\star_{12}} = (y^n)^{\star_{12}} = (y)$ *for all natural numbers* n;

$$(x^m)^{\star_{12}} = (x^m, y^n)^{\star_{12}} = (x^m + ay^n)^{\star_{12}} = (x^m, y)$$

for all natural numbers m *and* n *and all* $a \in k^\times$; $R^{\star_{12}} = R$.

(13) $\star_{13}$: $(0)^{\star_{13}} = (0)$; $(x^n)^{\star_{13}} = (x^n)$ *and* $(y^n)^{\star_{13}} = (y^n)$ *for all natural numbers* n; $I^{\star_{13}} = R$ *for all other* I.

(14) $\star_{14}$: $(0)^{\star_{14}} = (0)$; $(x^n)^{\star_{14}} = (x)$ *and* $(y^n)^{\star_{14}} = (y^n)$ *for all natural numbers* n; $I^{\star_{14}} = R$ *for all other* I.

(15) $\star_{15}$: $(0)^{\star_{15}} = (0)$; $(x^n)^{\star_{15}} = (x^n)$ *and* $(y^n)^{\star_{15}} = (y)$ *for all natural numbers* n; $I^{\star_{15}} = R$ *for all other* I.

(16) $\star_{16}$: $(0)^{\star_{16}} = (0)$; $(x^n)^{\star_{16}} = (x)$ *and* $(y^n)^{\star_{16}} = (y)$ *for all natural numbers* n; $I^{\star_{16}} = R$ *for all other* I.

(17) $\star_{17}$: $(0)^{\star_{17}} = (0)$; $(x^n)^{\star_{17}} = (x^n)$ *for all natural numbers* n; $I^{\star_{17}} = R$ *for all other* I.

(18) $\star_{18}$: $(0)^{\star_{18}} = (0)$; $(x^n)^{\star_{18}} = (x)$ *for all natural numbers* n; $I^{\star_{18}} = R$ *for all other* I.

(19) $\star_{19}$: $(0)^{\star_{19}} = (x^n)^{\star_{19}} = (x)$ *for all natural numbers* n; $I^{\star_{19}} = R$ *for all other* I.

(20) $\star_{20}$: $(0)^{\star_{20}} = (0)$; $(y^n)^{\star_{20}} = (y^n)$ *for all natural numbers* n; $I^{\star_{20}} = R$ *for all other* I.

(21) $\star_{21}$: $(0)^{\star_{21}} = (0)$; $(y^n)^{\star_{21}} = (y)$ *for all natural numbers* n; $I^{\star_{21}} = R$ *for all other* I.

(22) $\star_{22}$: $(0)^{\star_{22}} = (y^n)^{\star_{22}} = (y)$ *for all natural numbers* n; $I^{\star_{22}} = R$ *for all other* I.

(23) $\star_{23}$: $(0)^{\star_{23}} = (0)$; $I^{\star_{23}} = R$ *for all other* I.

(24) $\star_{24}$: $I^{\star_{24}} = R$ *for all* I.

Proof. Every non-unit, regular element in R has the form $x^n + ay^m$, $m, n \in \mathbb{N}$ and $a \in k^\times$. By computing $(((x^n + ay^m)I)^\star : x^n + ay^m)$ for $I = (0), I = (x^n), I = (y^n), I = (x^m + ay^n)$, and $I = (x^m, y^n)$ one can verify $(((x^n + ay^m)I)^\star : x^n + ay^m) = I^\star$ to show each closure from the above list is in fact standard.

Now we will show that these are in fact all the standard closures on R.

Suppose $\star$ is a standard closure and there exists an ideal I such that $I^\star \neq I$. If this is the case, the ideal I must be one of the following:

(a) $I = (0)$.
(b) $I = (x^n)$ for some natural number n.
(c) $I = (y^n)$ for some natural number n.

(d) $I = (x^n, y^m)$ for some natural numbers n and m.
(e) $I = (x^n + ay^m)$ for some natural numbers n and m and $a \in k^\times$.

(a) Suppose $I = (0)$. Either $(0)^\star = (x^n)$ for some natural number n, $(0)^\star = (y^n)$ for some natural number n, $(0)^\star = (x^n, y^m)$ for natural numbers n and m, $(0)^\star = (x^n + ay^m)$ for natural numbers n and m and some $a \in k^\times$, or $(0)^\star = R$.

If $(0)^\star = (x^n)$, then

$$(((x^r + ay^s)(0))^\star : x^r + ay^s) = (x^n : x^r + ay^s) = \begin{cases} (x) \text{ if } r \geq n-1 \\ (x^{n-r}) \text{ if } 1 \leq r < n-1. \end{cases}$$

Since $\star$ is standard, this contradicts $(0)^\star = (x^n)$ unless $(0)^\star = (x)$. Since $(0) \subseteq (x^n) \subseteq (x)$, then $(x^n)^\star = (x)$. Since $(0) \subseteq (y^m)$ then $(x) = (0)^\star \subseteq (y^m)^\star$. Thus $(x, y^m) \subseteq (y^m)^\star$. So either $(y^m)^\star = (x, y^i)$ for some $i \leq m$ or $(y^m)^\star = R$.

Note that if $(y^m)^\star = (x, y^i)$ then Lemma 3.12 allows us to conclude that $(y^m)^\star = (x, y^m)$. Also Lemma 3.11 implies that for all natural numbers m and n and all $a \in k^\times$, $(x^m + ay^n)^\star = (x, y^n)$. Clearly $R^\star = R$ and this is the operation described by $\star_7$.

If $(y^m)^\star = R$ for some m then by Lemma 3.8, $(y^m)^\star = (x^n + ay^m)^\star = (x^n, y^m)^\star = R$ for all $m, n \in \mathbb{N}$. This is the operation described by $\star_{19}$.

If $(0)^\star = (y^n)$ instead of (x^n) the same arguments used above exchanging the roles of x and y will produce the operations described by $\star_{12}$ and $\star_{22}$ respectively.

If $(0)^\star = (x^n, y^m)$ for some natural numbers n and m, then

$$\begin{aligned} (((x+y)(0))^\star : x+y) &= ((0)^\star : x+y) = ((x^n, y^m) : (x+y)) \\ &= \begin{cases} (x^{n-1}, y^{m-1}) \text{ if } n, m > 1 \\ (x, y^{m-1}) \text{ if } n = 1 \text{ and } m > 1 \\ (x^{n-1}, y) \text{ if } n > 1 \text{ and } m = 1 \\ R \text{ if } n = m = 1 \end{cases} \end{aligned}$$

exhibits that this closure is not standard so $(0)^\star \neq (x^n, y^m)$.

If $(0)^\star = (x^n + ay^m)$ for some natural numbers n and m and some $a \in k^\times$, then since

$$\begin{aligned} (((x+y)(0))^\star : x+y) &= ((0)^\star : x+y) = ((x^n + ay^m) : (x+y)) \\ &= \begin{cases} (x^{n-1} + ay^{m-1}) \text{ if } n, m > 1 \\ (x^n, y^m) \text{ if } n = 1 \text{ or } m = 1 \end{cases} \end{aligned}$$

exhibits that this closure is not standard so $(0)^\star \neq (x^n + ay^m)$.

If $(0)^\star = R$, then $I^\star = R$ for all I. Then $\star$ is standard and corresponds to $\star_{24}$.

For the remaining standard closures we may assume that $(0)^\star = (0)$.

(b) Suppose $I = (x^n)$. Then $(x^n)^\star = (x^i)$ for some natural number $i < n$, $(x^n)^\star = (x^i, y^j)$ for some natural number j and some $i \leq n$, $(x^n)^\star = (x^i + ay^j)$ for some natural number j and some $a \in k^\times$ and $i < n$, or $(x^n)^\star = R$.

Suppose first that $(x^n)^\star = (x^i)$ for some natural number $i < n$. By Lemma 3.7, then $(x^r)^\star = (x)$ for all natural numbers r. To determine $\star$ we need to determine $J^\star$ for the remaining ideals J of R. By Lemmas 3.6 and 3.7 if $(y^s)^\star = (y^i)$ for some $i \in \mathbb{N}$ either $(y^s)^\star = (y^s)$ or $(y^s)^\star = (y)$ for all $s \geq 1$.

Suppose $(y^s)^\star = (y^s)$ for all natural numbers s. Since $(x^r)^\star = (x)$ for all natural numbers r, $(x, y^s) = (x^r)^\star + (y^s)^\star \subseteq (x^r, y^s)^\star \subseteq (x, y^s)^\star$ for all natural numbers r and s. Applying $\star$ to the chain we obtain $(x^r, y^s)^\star = (x, y^s)^\star$ for all r and s. Either $(x, y)^\star = (x, y)$ and $(x, y)^\star = R$. Suppose the former. Since $x \in (x^r, y^s)^\star$ for all r and s by Lemma 3.10 $(x^r, y^s)^\star = (x, y^s)$ for r and s. Then by Lemma 3.11, $(x^r + ay^s)^\star = (x, y^s)$ for all natural numbers r and s and $a \in k^\times$ if $\star$ is standard. This corresponds to $\star_4$ above. If $(x, y)^\star = R$, by Lemma 3.9 we obtain $\star_{14}$.

Now suppose that $(y^s)^\star = (y)$ for all natural numbers s. Since $(x^r)^\star = (x)$ for all natural numbers r, $(x, y) = (x^r)^\star + (y^s)^\star \subseteq (x^r, y^s)^\star \subseteq (x, y)^\star$ for all r and s. Applying $\star$ to the chain we obtain $(x^r, y^s)^\star = (x, y)^\star$ for all r and s. If $(x, y)^\star = (x, y)$, then $\star$ cannot be standard since

$$(((x + y)(x, y))^\star : x + y) = ((x^2, y^2)^\star : x + y) = ((x, y) : x + y) = R.$$

Thus $(x, y)^\star = R$ and by Lemma 3.9, $\star$ corresponds to $\star_{16}$ above.

Now suppose that $(y^s)^\star = (x^i, y^j)$ for some natural numbers i and $j \leq s$. Then Lemma 3.12 implies that $(y^s)^\star = (x, y^s)$ for all s. Either $(x, y)^\star = (x, y)$ and $(x, y)^\star = R$. Suppose the latter. Then by Lemma 3.9 $(x, y^s)^\star = R$ for all s. This is a contradiction since (x, y^s) is $\star$ closed. Thus $(x, y)^\star = (x, y)$. Since

$$(x, y^s) = (y^s)^\star \subseteq (x^r, y^s)^\star \subseteq (x, y),$$

by Lemma 3.10 $(x^r, y^s)^\star = (x, y^s)$ for all s. Moreover, Lemma 3.11 implies that

$$(x^r + ay^s)^\star = (x, y^s)$$

for all natural numbers r and s and $a \in k^\times$. This corresponds to $\star_6$ above.

Now suppose that $(y^s)^\star = R$ for some $s \in \mathbb{N}$. Then Lemma 3.8 implies that

$$(y^s)^\star = (x^r, y^s)^\star = (x^r + ay^s)^\star = R$$

for all natural numbers r and s and all $a \in R^\times$. Thus $\star$ corresponds to $\star_{18}$ above.

We have not considered the case $(y^s)^\star = (x^n + ay^t)$ for some natural number n and $t < s$ and $a \in k^\times$. However by Lemma 3.13 this is not possible.

Now consider the cases that $(x^n)^\star = (x^i, y^j)$ for some natural numbers $i \leq n$ and j. Again Lemma 3.12 implies that $(x^r)^\star = (x^r, y)$ for all natural numbers r. Since

$$(x^r, y) = (x^r)^\star \subseteq (x^r, y^s) \subseteq (x^r, y)$$

for all natural numbers r and s, we see that $(x^r, y^s)^\star = (x^r, y)$ for all natural numbers r and s. Applying Lemma 3.11, we obtain $(x^r + ay^s)^\star = (x^r, y)$ for all natural numbers r and s and all $a \in k^\times$.

We need only determine the closures of (y^s) for all s and we will know what $\star$ is. If $(y^s)^\star = (y^s)$ for some s, then Lemma 3.6 tells us that $(y^s)^\star = (y^s)$ for all s. Thus $\star$ corresponds to $\star_{10}$. If $(y^s)^\star = (y^i)$ for some natural number $i < s$, then by Lemma 3.7 $(y^s)^\star = (y)$ for all s. Thus $\star$ corresponds with $\star_{11}$. Since $(y^s)^\star \subseteq (x^r, y^s)^\star = (x^r, y)$ for all natural numbers r and s, it cannot be the case that $(y^s)^\star = (x^i, y^j)$ for natural numbers i and j nor can $(y^s)^\star = R$.

By Lemma 3.13 we know it is not the case that $(x^m)^\star = (x^i + ay^j)$ for any natural numbers i, j and m or $a \in k^\times$.

Suppose $(x^n)^\star = R$ for some n. Then Lemma 3.8 implies that

$$(x^r)^\star = (x^r, y^s)^\star = (x^r + ay^s)^\star = R$$

for all natural numbers r and s and $a \in k^\times$. Either $(y^s)^\star = (y^s)$ for all natural numbers s, this is $\star_{20}$, or $(y^s)^\star = (y)$ for all natural numbers s which is $\star_{21}$ or $(y^s)^\star = R$ for all natural numbers s which is $\star_{23}$.

For the remaining cases we will assume that $(0)^\star = (0)$ and $(x^r)^\star = (x^r)$ for all natural numbers r.

(c) $I = (y^n)$. It is either the case that $(y^n)^\star = (y)$ for some natural number n, $(y^n)^\star = (x^i, y^j)$ for some natural numbers n,i and $j \leq n$, $(y^n)^\star = (x^i + ay^j)$ for some natural numbers n, i and $j < n$ and some $a \in k^\times$ or $(y^n)^\star = R$ for some natural number n.

Suppose first that $(y^n)^\star = (y)$ for some natural number n. Consider the case when $(x, y)^\star = R$. Then by Lemma 3.9 $(x^r, y^s)^\star = (x^r + ay^s)^\star = R$ for all $r, s \in \mathbb{N}$ and $a \in k^\times$. This is the operation described by $\star_{15}$.

Now consider the case when $(x, y)^\star = (x, y)$. By 3.10 either $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$, $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$. However we cannot have $(x^m, y^n)^\star = (x^m, y^n)$ since $(y) = (y^n)^\star \subseteq (x^m, y^n)^\star$ implies that $y \in (x^m, y^n)^\star$. Similarly we cannot have $(x^m, y^n)^\star = (x, y^n)$. So suppose $(x^m, y^n)^\star = (x^m, y)$ for all m, n. Then by 3.11 $(x^m + ay^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. This is the operation described by $\star_9$.

By Lemma 3.12 if $(y^n)^\star = (x^i, y^j)$ for some natural numbers i and $j \leq n$ then $(y^n) = (x, y^n)$ for all natural numbers n. Since $(x, y^n) = (y^n)^\star \subseteq (x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ by Lemma 3.11 $(x^m + ay^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. This is the operation described by $\star_5$.

By Lemma 3.13 $(y^n)^\star \neq (x^i + ay^j)$ for any natural numbers i and $j < n$ with $a \in k^\times$.

Lastly, consider the case that $(y^n)^\star = R$ for some natural number n. By Lemma 3.8

$$(y^r)^\star = (x^r, y^s)^\star = (x^r + ay^s)^\star = R$$

for all $r, s \in \mathbb{N}$ and $a \in k^\times$. This corresponds to the closure $\star_{17}$.

Let us assume for the remaining cases that $(0)^\star = (0)$, $(x^r)^\star = (x^r)$ and $(y^r)^\star = (y^r)$ for all natural numbers r.

(d) To determine $\star$ we need to determine $I^\star$ for the remaining ideals of R. We are assuming here that there are some natural numbers m and n such that $(x^m, y^n)^\star \neq (x^m, y^n)$. Consider first the case when $(x, y)^\star = R$. Then by Lemma 3.9 $(x^r, y^s)^\star = (x^r + ay^s)^\star = R$ for all $r, s \in \mathbb{N}$ and $a \in k^\times$. This is the operation described by $\star_{13}$.

Now consider the case when $(x, y)^\star = (x, y)$. By Lemma 3.10 since $(x^m, y^n)^\star \neq (x^m, y^n)$ for some $m, n \in \mathbb{N}$ then $(x^r, y^s)^\star = (x, y^s)$ for all $r, s \in \mathbb{N}$ or $(x^r, y^s)^\star = (x^r, y)$ for all $r, s \in \mathbb{N}$.

Suppose $(x^r, y^s)^\star = (x, y^s)$ for all $r, s \in \mathbb{N}$. Then by Lemma 3.11 $(x^r + ay^s)^\star = (x, y^s)$ for all $r, s \in \mathbb{N}$ and $a \in k^\times$. This is the operation described by $\star_3$.

Suppose $(x^r, y^s)^\star = (x^r, y)$ for all $r, s \in \mathbb{N}$. Then by Lemma 3.11 $(x^r + ay^s)^\star = (x^r, y)$ for all $r, s \in \mathbb{N}$ and $a \in k^\times$. This is the operation described by $\star_8$.

(e) At this point $(x^r)^\star = (x^r)$ for all r and $(y^r)^\star = (y^r)$ for all r and $(x^r, y^s)^\star = (x^r, y^s)$ for all $r, s \in \mathbb{N}$. Since $(x^m + ay^n)^\star \neq (x^m + ay^n)$ for some $m, n \in \mathbb{N}$ and $a \in k^\times$ and $(x^m + ay^n)^\star \subseteq (x^m, y^n)^\star = (x^m, y^n)$ we have $(x^m + ay^n)^\star = (x^m, y^n)$. Let u, $v \in \mathbb{N}$ and $b \in k^\times$. Since $\star$ is weakly prime we have

$$(x^u + \frac{b}{a}y^v)(x^m + ay^n)^\star \subseteq ((x^u + \frac{b}{a}y^v)(x^m + ay^n))^\star$$

which implies

$$(x^{m+u}, y^{n+v}) \subseteq (x^{m+u} + by^{n+v})^\star$$

and

$$(x^{m+u} + by^{n+v})^\star = (x^{m+u}, y^{n+v})^\star = (x^{m+u}, y^{n+v})$$

Thus $(x^r + by^s)^\star = (x^r, y^s)$ for all $b \in k^\times$, $r > m$ and $s > n$. Now suppose r, $s \in \mathbb{N}$. We have

$$\begin{aligned} ((x^m + y^n)(x^r + by^s))^\star : x^m + y^n) &= ((x^{m+r} + by^{n+s})^\star : x^m + y^n) \\ &= ((x^{m+r}, y^{n+s}) : x^m + y^n) = (x^r, y^s). \end{aligned}$$

Since $\star$ is standard this implies $(x^r + by^s)^\star = (x^r, y^s)$ for all $b \in k^\times$ and r, $s \in \mathbb{N}$. This is the operation described by $\star_2$.

The only closure that we are now missing is the identity $\star_1$. $\square$

Remark 3.15. Due to Theorem 3.4 we conclude that there are 24 finite type semistar operations on $k[[x, y]]/(xy)$. The only one which is a star operation is the identity.

Note that if we only consider the nine standard closure operations $\star_1, \star_7, \star_{12}, \star_{17}, \star_{19}, \star_{20}, \star_{22}, \star_{23}$ and $\star_{24}$, all the remaining standard closures can be obtained from these through intersections.

- $\star_2 = \star_7 \cap \star_{12} \cap \star_{17} \cap \star_{20}$.
- $\star_3 = \star_7 \cap \star_{17} \cap \star_{20}$.
- $\star_4 = \star_7 \cap \star_{20}$.
- $\star_5 = \star_7 \cap \star_{17}$.
- $\star_6 = \star_7 \cap \star_{23}$.
- $\star_8 = \star_{12} \cap \star_{17} \cap \star_{20}$.
- $\star_9 = \star_{12} \cap \star_{17}$.
- $\star_{10} = \star_{12} \cap \star_{20}$.
- $\star_{11} = \star_{12} \cap \star_{23}$.
- $\star_{13} = \star_{17} \cap \star_{20}$.
- $\star_{14} = \star_{19} \cap \star_{20}$.
- $\star_{15} = \star_{17} \cap \star_{22}$.
- $\star_{16} = \star_{19} \cap \star_{22}$.
- $\star_{18} = \star_{19} \cap \star_{23}$.
- $\star_{21} = \star_{22} \cap \star_{23}$.

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