Pigeonhole Principle
for Janet Vassilev's Math 327 course

Let $I_n = \{1, 2, \dots, n\}$ for any natural number n.

Theorem 0.1 (Pigeonhole Principle) *Suppose we have $n > m$ objects that need to be placed in m boxes. Then at least one box has at least two objects in it.*

Proof: Let I_n represent the n objects and I_m represent the m boxes. Consider a function $f : I_n \to I_m$ which represents the placement of objects into boxes. Since $n > m$ then f cannot be one to one so there exists $i, j \in I_n$ and $k \in I_m$ with $f(i) = f(j) = k$. Thus the box represented by k has at least two objects placed in it.

Example 0.2 *Suppose $a_1, \dots, a_8$ are all integers. Show that at least two of them have the same remainder when divided by 7.*

Since there are precisely 7 remainders 0, 1, 2, 3, 4, 5, 6, we take these remainders to be the boxes. We place a_i into a box represented by the remainder r if $a_i \equiv r \mod 7$. Since there are 8 integers, then by the pigeonhole principle, at least one of the remainders has two integers in it. Thus, these two have the same remainder when divided by 7.

Example 0.3 *Suppose nine points are to be placed in a cube with sides 2 centimeters. Show that at least two points are within $\sqrt{3}$ centimeters.*

Divide the cube into 8 smaller cubes with sides 1 centimeter. Since the diagonal of such a cube is $\sqrt{3}$, then any two points within this cube will have distance at most $\sqrt{3}$ centimeters. Since there are 9 points, then at least one of the smaller cubes has two points within it by the pigeonhole principle. Thus, at least two of the points are within $\sqrt{3}$ centimeters.

Example 0.4 *Suppose $a_1, \dots, a_8$ are all integers. Show that some set of consecutive numbers in the list has a sum divisible by 8.*

Consider $b_1 = a_1$, $b_2 = a_1 + a_2, \dots b_8 = a_1 + a_2 + \cdots + a_8$. If any of these is divisible by 8 we are done. So suppose none are divisible by 8. This means that all of the b_i have some positive remainder when divided by 8. Since there are precisely 7 positive remainders 1, 2, 3, 4, 5, 6, 7, we take these remainders to be the boxes. We place b_i into a box represented by the remainder r if $a_i \equiv r \mod 8$. Since there are 8 sums, then by the pigeonhole principle, at least one of the remainders has two sums assigned to it. Thus, these two have the same remainder when divided by 8. However, if $b_i - b_j = a_{j+1} + \cdots a_i$ which is a sum of consecutive integers and we are done.

Theorem 0.5 (Strong Pigeonhole Principle) *Suppose we have $n > m$ objects that need to be placed in m boxes. Then at least one box has at least $\lceil \frac{n}{m} \rceil$ objects in it.*

Example 0.6 *Suppose 25 points are to be placed in a square with sides 2 centimeters, show that at least 7 points are within a distance of $\sqrt{2}$ centimeters.*

The square can be divided into 4 squares each with sides 1 centimeter. Since $\lceil \frac{25}{4} \rceil = 7$ then by the Strong Pigeonhole Principle, one of the squares has at least 7 points inside. Since the maximum distance between points in this square is $\sqrt{2}$ centimeters, then there are 7 points within $\sqrt{2}$ centimeters.