The Recurrence Relations
for Janet Vassilev's Math 327 course

Suppose we have a function $f : \mathbb{N} \to \mathbb{R}$. Setting $a_n = f(n)$ for all $n \in \mathbb{N}$, we term the set $\{a_n\}_{n=1}^{\infty}$ a *sequence*. Suppose we know $a_1, \ldots, a_k$ and for $a_n = f(a_{n-1}, \ldots, a_{n-k})$ for some function $f : \mathbb{R}^k \to \mathbb{R}$, we say $\{a_n\}_{n=1}^{\infty}$ is a *recursively defined sequence* given by the *recurrence relation* $a_n = f(a_{n-1}, \ldots, a_{n-k})$.

We say a recurrence relation is *linear* if f is a linear function or in other words, $a_n = f(a_{n-1}, \ldots, a_{n-k}) = s_1 a_{n-1} + \cdots + s_k a_{n-k} + f(n)$ where $s_i, f(n)$ are real numbers. A linear recurrence relation is *homogeneous* if $f(n) = 0$.

The *order* of the recurrence relation is determined by k. We say a recurrence relation is of *order k* if $a_n = f(a_{n-1}, \ldots, a_{n-k})$. We will discuss how to solve linear recurrence relations of orders 1 and 2.

1 Homogeneous linear recurrence relations

Let $a_n = s_1 a_{n-1}$ be a first order linear recurrence relation with $a_1 = k$. Notice, $a_2 = s_1 k$, $a_3 = s_1 a_2 = s_1^2 k$, $a_4 = s_1 a_3 = s_1^3 k$, and in general $a_n = k s_1^{n-1}$.

Example 1.1 *If $a_1 = 4$ and $a_n = \dfrac{a_{n-1}}{2}$ for $n \geq 2$, then $a_n = 4(\frac{1}{2}^{n-1}) = \frac{1}{2^{n-3}}$.*

Suppose now that we have a homogeneous linear recurrence relation of order 2: $a_n = s_1 a_{n-1} + s_2 a_{n-2}$ with $a_1 = k_1$ and $a_2 = k_2$. We take a guess that the solution will be of the form $a_n = cr^n$. Substituting this into our recurrence relation we obtain

$$cr^n = s_1 cr^{n-1} + s_2 cr^{n-2}.$$

Factoring out cr^{n-2} we obtain a quadratic equation: $r^2 = s_1 r + s_2$ or

$$r^2 - s_1 r - s_2 = 0.$$

We have three possibilities for the roots of this quadratic equation: two distinct real roots a and b, a unique double root a or two complex conjugate roots $a + ib$ and $a - ib$. The solutions to the recurrence relation will depend on these roots of the quadratic equation.

Suppose first that the recurrence relation has two distinct real roots a and b, then the solution of the recurrence relation will be $a_n = c_1 a^n + c_2 b^n$. We use $a_1 = k_1$ and $a_2 = k_2$ to solve the recurrence relation. Since these give us values to solve a system of equations in two variables c_1 and c_2:

$$k_1 = c_1 a + c_2 b$$
$$k_2 = c_1 a^2 + c_2 b^2.$$

Example 1.2 *Let $a_1 = 3$ and $a_2 = 7$ and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. The corresponding quadratic equation is $r^2 - 2r - 3 = 0$ which has roots 3 and -1. So our solution should have the form $a_n = c_1 3^n + c_2(-1)^n$. We must now solve the system of equations*

$$3 = 3c_1 - c_2$$

$$7 = 9c_1 + c_2.$$

Adding the two equations together we obtain $10 = 12c_1$ *or* $c_1 = \frac{5}{6}$. *So* $c_2 = 3(\frac{5}{6}) - 3 = -\frac{1}{2}$. *So our solution is*

$$a_n = \frac{5}{6}3^n - \frac{1}{2}(-1)^n$$

or

$$a_n = \frac{5}{2}3^{n-1} + \frac{1}{2}(-1)^{n-1}.$$

If our recurrence relation has a unique double root a, then our solution will have the form $a_n = (c_1 + c_2 n)a^n$ this distinguishes us from the order one case since each quadratic has two roots. Again we use the k_1 and k_2 to set up a system of equations in c_1 and c_2 to find the solution of a_n.

Example 1.3 *Let* $a_1 = 2$ *and* $a_2 = 5$ *and* $a_n = 6a_{n-1} - 9a_{n-2}$ *for* $n \geq 3$. *The corresponding quadratic equation is* $r^2 - 6r + 9 = 0$ *which has a unique double root* 3. *So our solution should have the form* $a_n = (c_1 + nc_2)3^n$. *We must now solve the system of equations*

$$2 = 3c_1 + 3c_2$$

$$5 = 9c_1 + 18c_2.$$

Subtracting 3 times the first equation from the second we obtain $-1 = 9c_2$ *or* $c_2 = -\frac{1}{9}$. *So* $c_1 = \frac{1}{9} + \frac{2}{3} = \frac{7}{9}$. *So our solution is*

$$a_n = \frac{7}{9}3^n - \frac{1}{9}n3^n$$

or

$$a_n = 7(3^{n-2}) + -n(3^{n-2}).$$

If our recurrence relation has two complex conjugate roots, we could write our solution the way we did in the case where we had two real roots: $a_n = c_1(a + bi)^n + c_2(a - bi)^n$. However, there is a more compact way to write our solution in terms of real numbers. We can write $a + bi = re^{i\theta}$ where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$. Then $a - bi = re^{-i\theta}$. Using DeMoivre's Theorem $(re^{i\theta})^n = r^n(\cos n\theta + i \sin n\theta)$. Thus

$$c_1(re^{i\theta})^n + c_2(re^{-i\theta})^n = r^n[(c_1 + c_2)\cos n\theta + i(c_1 - c_2)\sin n\theta].$$

If we set $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$, then our solution is

$$a_n = r^n(C_1 \cos n\theta + C_2 \sin n\theta).$$

Again we can use k_1 and k_2 to solve a system of equations in C_1 and C_2.

Example 1.4 *Let* $a_1 = 1$ *and* $a_2 = 2$ *and* $a_n = a_{n-1} - a_{n-2}$. *The corresponding quadratic equation is* $r^2 - r + 1 = 0$ *which has roots* $\frac{1 \pm i\sqrt{3}}{2}$. *Note that* $r = 1$ *and* $\theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$. *So our solution should have the form* $a_n = C_1 \cos\frac{n\pi}{3} + C_2 \sin\frac{n\pi}{3}$. *We must now solve the system of equations*

$$1 = \frac{C_1}{2} + \frac{C_2\sqrt{3}}{2}$$

$$2 = -\frac{C_1}{2} + \frac{C_2\sqrt{3}}{2}.$$

Adding the equations together we obtain $3 = C_2\sqrt{3}$ *or* $C_2 = \sqrt{3}$. *So* $C_1 = 2 - 3 = -1$. *So our solution is*

$$a_n = -\cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}.$$

2 Nonhomogeneous linear recurrence relations

When $f(n) \neq 0$, we will search for a particular solution a_n^p which is similar to $f(n)$. We will still solve the homogeneous recurrence relation setting $f(n)$ temporarily to 0 and the solution of this homogeneous recurrence relation will be a_n^h and $a_n = a_n^p + a_n^h$. The following table provides a good first guess:

$f(n)$	a_n^p
$a_0 + a_1 n + \cdots + a_r n^r$	$b_0 + b_1 n + \cdots + b_r n^r$
ar^n	br^n
$a\cos n\theta + b\sin n\theta$	$c\cos n\theta + d\sin n\theta$

However, if the solutions to the related homogeneous recurrence relation are similar to your function $f(n)$ then you must multiply by an appropriate power of n. When solving nonhomogeneous recurrence relations, you need to find the particular solution first. Then you will solve for the needed coefficients of your related homogeneous solution using the initial data. This will be illustrated in the examples below.

Example 2.1 *Suppose* $a_n = 2a_{n-1} - 1$ *for* $n \geq 2$ *and* $a_1 = 3$. *Since* $f(n) = -1$ *and* $a_n^h = c2^n$, *then we will guess that* $a_n^p = b$. *We now plug this into the recurrence relation to solve for* b. *Since* $b = 2b - 1$ *we see that* $a_n^p = 1$. *Now we can solve for the* c. *Since* $a_1 = 3 = 2c + 1$ *we see that* $c = 1$ *and* $a_n = 2^n + 1$.

Example 2.2 *Suppose* $a_n = 2a_{n-1} - 2^n$ *for* $n \geq 2$ *and* $a_1 = 3$. *Since* $f(n) = -2^n$ *and* $a_n^h = c2^n$, *then we will guess that* $a_n^p = bn2^n$. *We now plug this into the recurrence relation to solve for* b. *Since* $bn2^n = 2b(n-1)2^{n-1} - 2^n$ *we see that* $bn = b(n-1) - 1$ *or* $b = -1$. *Thus* $a_n^p = -n2^n$. *Now we can solve for the* c. *Since* $a_1 = 3 = 2c - 2$ *we see that* $c = \frac{5}{2}$ *and* $a_n = 5(2^{n-1}) - n2^n$.

Note if we did not choose our particular solution to be $bn2^n$ but $b2^n$ in the above example, then we would get $b2^n = b2^n - 2^n$ or $0 = -2^n$ and we cannot solve for b.

Example 2.3 *Suppose* $a_n = 2a_{n-1} - a_{n-2} + 2$ *for* $n \geq 3$ *with* $a_1 = 1$ *and* $a_2 = 5$. *Since* $f(n) = 2$ *and* $a_n^h = c_1 + c_2 n$, *then we will guess that* $a_n^p = bn^2$. *We now plug this into the recurrence relation to solve for* b. *Since* $bn^2 = 2b(n-1)^2 - b(n-2)^2 + 2$ *we see that* $bn^2 = 2bn^2 - 4bn + 2b - bn^2 + 4bn - 4b + 2$ *or* $2b = 2$ *implying* $b = 1$. *Thus* $a_n^p = n^2$ *Now we can solve for the* c_1 *and* c_2. *Since* $a_1 = 1 = c_1 + c_2 + 1$ *and* $a_2 = 5 = c_1 + 2c_2 + 4$, *we obtain* $c_1 + c_2 = 0$ *and* $c_1 + 2c_2 = 1$ *and see that* $c_2 = 1$ *and* $c_1 = -1$ *and* $a_n = -1 + n + n^2$.

Example 2.4 *Suppose* $a_n = a_{n-1} + \sin\frac{n\pi}{2}$ *for* $n \geq 2$ *and* $a_1 = -1$. *We guess that* $a_n^p = a\sin\frac{n\pi}{2} + b\cos\frac{n\pi}{2}$ *since* $a_n^h = c$. *We need to solve for* a *and* b *so we plug the particular solution into the recurrence relation.* $a\sin\frac{n\pi}{2} + b\cos\frac{n\pi}{2} = a\sin\frac{(n-1)\pi}{2} + b\cos\frac{(n-1)\pi}{2} + \sin\frac{n\pi}{2}$. *Using the trig identities* $\sin(A+B) = \sin A\cos B + \sin B\cos B$ *and* $\cos(A+B) = \cos A\cos B - \sin A\sin B$, *we obtain*

$$\begin{aligned} a\sin\frac{n\pi}{2} + b\cos\frac{n\pi}{2} &= a(\sin\frac{(n)\pi}{2}\cos(\frac{\pi}{2}) - \cos\frac{n\pi}{2}(\sin\frac{\pi}{2})) + b(\cos\frac{n\pi}{2}\cos\frac{\pi}{2} + \sin\frac{n\pi}{2}\sin\frac{\pi}{2}) + \sin\frac{n\pi}{2} \\ &= -a\cos\frac{n\pi}{2} + b\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} \end{aligned}$$

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Simplifying we get

$$(a-b)\sin\frac{n\pi}{2}+(a+b)\cos\frac{n\pi}{2}=\sin\frac{n\pi}{2}$$

which implies that $a-b=1$ *and* $a+b=0$ *so* $a=\frac{1}{2}$ *and* $b=-\frac{1}{2}$ *and* $a_n^p=\frac{1}{2}\sin\frac{n\pi}{2}-\frac{1}{2}\cos\frac{n\pi}{2}$. *Now we solve for* c *using* $a_1=-1$ *and* $-1=c+\frac{1}{2}$ *implies that* $c=-\frac{3}{2}$ *So*

$$a_n=-\frac{3}{2}+\frac{1}{2}\sin\frac{n\pi}{2}-\frac{1}{2}\cos\frac{n\pi}{2}.$$