Math 421 – Spring 2014
Homework 5

1. What are the degrees of the following fields over $\mathbb{Q}$?

   (a) $\mathbb{Q}(\sqrt[3]{p})$ for any prime p.

   (b) $\mathbb{Q}(\alpha)$ where α is a root of the polynomial $x^3 - x + 1$.

   (c) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

   (d) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

2. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

3. Let F be a field and $a, b \in F$. Let $\alpha^2 = a$ and $\beta^2 = b$. Assume that α, β have degree 2 over F. Prove that $F(\alpha) = F(\beta)$ if and only if there exists $c \in F$ such that $a = c^2 b$.

4. Suppose that $[F(\alpha) : F]$ is odd. Show that $F(\alpha) = F(\alpha^2)$.

5. Let α and β be algebraic over a field F. Suppose $deg(\alpha, F)$ and $deg(\beta, F)$ are relatively prime. Show $irr(\beta, F)$ is irreducible over $F(\alpha)$.

6. Suppose $F \subseteq E \subseteq K$ is a chain of fields such that E is algebraic over F and K is algebraic over E. Show that K is algebraic over F.

7. Let E be an extension field of F and D be a domain with $F \subseteq D \subseteq E$. Show that D is a field.

8. Show that if $[E : F] = p$ for some prime p, then there are no proper extensions of F contained in E.