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Test Ideals in Gorenstein isolated singularities and F-finite reduced rings

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by

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DEDICATION

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PUBLICATIONS AND PRESENTATIONS

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ABSTRACT OF THE DISSERTATION

Test Ideals in Gorenstein isolated singularities and F-finite reduced rings

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Professor Richard Elman, Chair

The focal point of this thesis is an object called the test ideal defined in 1.14. Test ideals are defined in commutative rings which contain a field. In particular, we will restrict our study to commutative rings which contain a field of characteristic p. Just over ten years ago, Hochster and Huneke developed tight closure, which focuses on characteristic p methods in commutative algebra. Test ideals have become an important object to study in tight closure in part because of their link to the singularity type or the commutative ring. This thesis focuses on studying aspects of the test ideal in commutative rings of characteristic p which are: 1) F-finite (defined in 1.16) reduced quotients of a regular local ring and have F-pure singularities defined in 1.22 and 2) Gorenstein local domains.

One of our main results is that if R is F-pure and τ is the test ideal, R/τ is also

F-pure. Using this result, we can form a filtration of R, $(0) \subseteq \tau = \tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_i \subseteq \ldots \tau_c$ such that R/τ_i is F-pure, its test ideal is τ_{i+1}/τ_i and R/τ_c is F-regular. This filtration gives us some insight as to what power of the tight closure of an ideal is contained in the ideal itself. For example, for any ideal J in R with $J \subseteq \tau_c$ then $(J^*)^{h+1} \subseteq J$ where $h = \max\{\mathrm{ht}(P_i) | P_i \subseteq \mathrm{Ass}(J)\}$.

We will classify all complete Gorenstein local domains with test ideal equal to the maximal ideal which are one-dimensional and two-dimensional with the property that the square of the maximal ideal is contained in any minimal reduction of the maximal ideal. We show that the only complete one-dimensional local domain with test ideal equal to the maximal ideal is $k[[x^2, x^3]]$. In the two dimensional case, we have a complete classification of all complete two-dimensional Gorenstein local domains of the form $R = k[[x, y, z]]/(x^2 - a)$ with test ideal equal to the maximal ideal.

Introduction

Let R be a commutative Noetherian ring of characteristic $p > 0$ and I be an ideal of R with generators $x_1, \ldots, x_n$. Denote powers of p by q. Define R^o to be the complement of the union of minimal primes. We say $x \in R$ is in the tight closure of I if there exists a $c \in R^o$ such that $cx^q \in I^{[q]}$ for all large q, where $I^{[q]} = (x_1^q, \ldots, x_n^q)$. Denote the tight closure of I as I^*. In the definition of tight closure above we observe that the element c must be chosen in R^o. It is not necessarily the case that if $cx^q \in I^{[q]}$ for all large q and all $x \in I^*$, then for any other ideal J in R and $y \in J^*$ that $cy^q \in J^{[q]}$ for all large q. However, if for all ideals $J \subseteq R$, $c(J^*)^{[q]} \subseteq J^{[q]}$ for all q we call c a test element. The ideal generated by these test elements is called the test ideal.

Since Hochster and Huneke first defined tight closure several others have collaborated to solve many problems in both commutative algebra and algebraic geometry including: the "homological conjectures", Big Cohen Macaulay Modules, singularity theory, the Briançon-Skoda Theorem and more. Test elements and test ideals play a key role in tight closure theory. Knowing test elements allows us to compute tight closures of ideals and prove persistence of tight closure which we state later in Theorem 1.18. Also in the case where R is a Gorenstein isolated singularity (defined in 1.4), the singularity type of the ring R can be determined by the test ideal. My thesis involves the study of test ideals in two situations: 1) when the ring is F-finite and a reduced quotient of a regular local ring, and 2) when the ring is a Gorenstein isolated singularity.

Recall a ring R is F-finite if $F(R)$ is a finite R-module. Let S be an F-finite regular local ring and I an ideal contained in S. Define $R = S/I$. A special class of reduced quotients of regular local rings in which we are interested are those which are F-pure, defined as follows. A monomorphsim $\varphi : R \to S$ is said to be pure if

$$\varphi \bigotimes 1_M : R \bigotimes M \to S \bigotimes M$$

is injective for all R-modules M. A reduced ring is said to be F-pure if the Frobenius homomorphism is pure. We require R to be reduced otherwise, the Frobenius homomorphism need not be injective. In [5], Fedder gave a nice criterion to check whether a quotient of a regular local ring $R = S/I$ is F-pure. It states such a ring is F-pure if and only if $(I^{[p]} :_S I) \not\subseteq \mathfrak{m}^{[p]}$. We want to understand $(I^{[q]} :_S I)$ further. Even if $I \subseteq J$ we don't necessarily have that $(I^{[q]} :_S I) \subseteq (J^{[q]} :_S J)$. But in such a ring we have shown that $(I^{[q]} :_S I) \subseteq (\tau_R^{[q]} :_S \tau_R)$ where τ_R is the pull back of the test ideal for R. In proving this, we have noted a new proof for Fedder's criterion. If R is F-pure then $(I^{[p]} :_S I) \not\subseteq \mathfrak{m}^{[p]}$ implies that $(\tau_R^{[p]} :_S \tau_R) \not\subseteq \mathfrak{m}^{[p]}$ as a consequence of the above theorem.

For all minimal primes P/I in R we know that $R_{P/I} = (S/I)_{P/I} = S_P/P$ which is a field. Since the regular locus is open in an excellent local ring we can conclude that it is nonempty. Thus if we choose c to be an element such that the primes not containing c are contained inside the regular locus then R_c is regular and thus some power is in the test ideal (defined in 1.17). Since F-pure rings are reduced then c is not nilpotent so it follows that $\mathrm{ht}((c)) = 1$. Therefore, $\mathrm{ht}(\tau) \geq 1$. Define $R_1 = S/\tau_R$ and $I \subsetneq \tau_R$ in S.

Applying Fedder's F-purity criterion we see that R_1 is F-pure. Since R_1 is defined to be a reduced quotient of a regular local ring, we note that S/τ_{R_1}, where τ_{R_1}, the pullback of the test ideal in S, is also F-pure. We can continue in this fashion defining $R_i = S/\tau_{R_{i-1}}$ where $\tau_{R_{i-1}}$ is the pullback of the test ideal of R_{i-1} in S as long as R_{i-1} is not F-regular; otherwise $S/\tau R_{i-1} \cong 0$. Now τ_{R_i}/I define a unique filtration of R such that $R/(\tau_{R_i}/I) \cong R_{i+1}$ is F-pure with test ideal $\tau_{R_{i+1}}/I$ and $(\tau_{R_{i+1}}/I)/(\tau_{R_i}/I)$ have positive height.

To find examples of these filtrations I have made explicit calculations of test ideals in polynomial and power series rings modulo monomial ideals. We see in this case that the test ideal is given by the following theorem:

Theorem 0.1 *Let $R = T/I$ where T is either a polynomial or power series ring and $I = P_1 \bigcap \ldots \bigcap P_n$ is generated by monomials and R/P_i are regular. Set $J = \Sigma(P_1 \bigcap \ldots \bigcap \hat{P_i} \bigcap \ldots \bigcap P_n)$, where $\hat{P_i}$ means P_i is not included in the intersection. Then $J = \tau$.*

To make this more explicit for ideals generated by all $\binom{n}{i}$ square-free monomials with i elements I have shown:

Example 0.2 *Let $T = k[[x_1, \ldots, x_n]]$ or $T = k[x_1, \ldots, x_n]$. Set $R = T/I$ where*

$$I = < x_{i_1} \ldots x_{i_d} >_{1 \leq i_1 < \ldots < i_d \leq n}$$

then $\tau = < x_{i_1} \ldots \hat{x_{i_r}} \ldots x_{i_d} >_{1 \le i_1 < \ldots < i_d \le n}$ *where* $\hat{x_{i_r}}$ *means* x_{i_r} *is not included in the product.*

Goto and Watanabe classify all 1 dimensional F-pure rings in [10]. One might wonder if such a filtration is a possible classifying tool for n dimensional F-pure rings where $n \ge 1$.

Another way to utilize the filtration of successive test ideals in F-pure rings is to get a bound for n such that $(I^*)^n \subseteq I$. To make the filtration easier to work with we reformulate the filtration by replacing i with $c - i$ in the above to get the following filtration, $\tau_c \subseteq \tau_{c-1} \subseteq \ldots \subseteq \tau_1 \subseteq \tau_0$ such that τ_i/τ_{i+1} is the test ideal for R/τ_{i+1}, the height of the successive test ideals are all greater than or equal to 1 and R/τ_i are all F-pure. We can formulate the following theorem:

Theorem 0.3 *Let* $R = S/I$, S *an* F*-finite regular local ring. Suppose that* R *is* F*-pure and* J/I *is an ideal of* R. *Let* $c+1$ *be the length of the unique filtration described above. For some* $t \in \{0, 1, \ldots, c\}$, $J \subseteq \tau_i$ *for* $i \le t$ *and if* $\tau_t \not\subseteq Ass(J)$ *then* $(J^*)^{c-t+1} \subseteq J$.

Since $\text{ht}(\tau_i) > \text{ht}(\tau_{i+1})$, for all i we get the following corollary:

Theorem 0.4 *Let* $R = S/I$, S *an* F*-finite regular local ring. Suppose that* R *is* F*-pure. If* $J \subseteq \tau_0$ *then* $(J^*)^{h+1} \subseteq J$ *where* $h = max\{ht(P_i) | P_i \subseteq Ass(J)\}$.

In local Gorenstein rings the parameter test ideal is precisely the test ideal. For general Cohen-Macaulay rings this is not the case, and furthermore, the tight closure of parameter ideals is not as well understood. For example, we can show using concepts which Karen Smith defines in [28] that over Gorenstein rings

$$(x_1, \ldots, x_d) : \tau = (x_1, \ldots, x_d)^*.$$

This statement is not known for Cohen-Macaulay rings. However, the above is true for one dimensional Cohen-Macaulay domains, but the proof heavily relies on integral closure since the test ideal is the conductor for 1 dimensional domains. Using this result, we have classified one dimensional complete Gorenstein domains with test ideal equal to the maximal ideal.

A main thrust in understanding test elements and test ideals in Gorenstein isolated singularities is to try to link the type of singularity with the test ideal. Recently Hara [11] used geometric methods to show that this tie exists in characteristic 0. Of current interest is classifying Gorenstein domains with test ideal equal to the maximal ideal. We show these are exactly the minimally elliptic singularities classified by Laufer [25]. Our proof relies strictly on tight closure methods.

We use the tight closure Briançon-Skoda theorem, which follows, to break down the classification of normal Gorenstein domains with test ideal equal to the maximal ideal into double points and triple points.

Theorem 0.5 *Let R a Noetherian local ring of characteristic p. Let I be an ideal in R generated by n elements. Then*

$$\overline{I^{n+r}} \subseteq (I^{r+1})^*.$$

Thus if $d=2$ and y,z is a system of parameters which is a minimal reduction of the maximal ideal then the above theorem says $\mathfrak{m}^2 \subseteq (y,z)^*$. Note $\tau\mathfrak{m}^2 \subseteq (y,z)$. If $\tau = \mathfrak{m}$ then $\mathfrak{m}^3 \subseteq (y,z)$. But $\mathfrak{m}^2 \subseteq (y,z)$ is possible so we break down the classification down into the two cases: 1) $\mathfrak{m}^2 \subseteq (y,z)$ and 2) $\mathfrak{m}^3 \subseteq (y,z)$ where $\mathfrak{m}^2 \not\subseteq (y,z)$. The author classifies case 1) using the methods below. First one shows that a Gorenstein normal domain satisfying 1) is isomorphic to a hypersurface with a double point. Thus we can reduce the problem of finding the power series representing this hypersurface to an element a that lies strictly in $k[[y,z]]$. Working in this context, we show that the isomorphism classes will be determined by

$$a \in (y,z)^4 \text{ or } a \text{ in the form } y^3 + b$$

such that $b \in (yz^4) + (y,z)^6$ where $y^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ and $z^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ in $k[[y,z]]$. To examine these a we use the following theorem.

Theorem 0.6 *[21] Let $(S,\mathfrak{m})$ be an equidimensional two-dimensional graded ring over a perfect field $k = S_0$ or characteristic $p >> 0$. Assume S_P regular for $P \subsetneq \mathfrak{m}$. Then for any system of parameters y,z of S with the sum of the degrees equal to δ, then*

$$(y,z)^* = (y,z) + S_{\geq\delta}.$$

When f is quasihomogeneous we can use the above theorem to compute $(y^2,z^2)^*$. To compute which power series are in the same isomorphism class as some polynomial f we use the Weierstrass Preparation Theorem 3.30.

CHAPTER 1

Preliminaries

1.1 Commutative Ring Theory

Let R be a commutative Noetherian ring. We define the dimension of R as follows:

Definition 1.1 *Let R be a commutative Noetherian ring. The* **dimension** *of R, sometimes called the Krull dimension of R, equals*

$$sup\{n | P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n, \text{ where the } P_i \text{ are prime ideals.}\}$$

There are many structures inside a commutative ring and many types of commutative rings that we must familiarize ourselves with.

If I and J are ideals in a commutative Noetherian ring R, we denote the ideal generated by the elements multiplying J into the ideal I by $(I :_R J) = \{x \in R | xJ \subseteq I\}$. If R is understood, we will supress the subscript and write it as $(I : J)$.

Definition 1.2 *Let $(R, \mathfrak{m})$ be a local commutative Noetherian ring. The* **socle** *of R is the ideal defined by $(0 : \mathfrak{m})$.*

A non-unit ideal I is said to be a **primary** ideal if for every product $ab \in I$ such that $a \notin I$ then $b^n \in I$ for some integer n. The radical of a primary ideal is a prime ideal called the associated prime. The proof of this fact can be found in Matsumura [27, 6.6]. To avoid confusion we will always denote a primary ideal associated to a prime ideal P as a P-primary ideal.

Let $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_n = 0$ be a filtration of M such that $M_i/M_{i+1} \cong R/m_i$ where m_i is a maximal ideal in R. If such a filtration exists then n is called the **length** of M, and denote the length $l(M)$. For an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ the **multiplicity** of $\mathfrak{q}$ denoted $e(\mathfrak{q}) = \lim_{n \to \infty} \frac{d!}{n^d} l(R/\mathfrak{q}^n)$ where d is the dimension of R.

In a d-dimensional local ring $(R, \mathfrak{m})$, if $x_1, \ldots, x_d$ generate an $\mathfrak{m}$-primary ideal, we say $x_1, \ldots, x_d$ is a **system of parameters**. If I and J are ideals such that $I \subseteq J$, then we say I is a **reduction** of J if there exists an integer n such that $IJ^n = J^{n+1}$. The least such n such that equality holds for all reductions J is called the **reduction number**. Reductions are often much nicer to work with because they can have fewer generators.

Definition 1.3 *We say J is a **minimal reduction** of an ideal I if we cannot find an ideal $K \subsetneq J$ such that K is a reduction of I.*

See Vasconcelos for more details on reductions [31, Chapter 5]. Also for any reduction of an $\mathfrak{m}$-primary ideal we know that the multiplicities are necessarily equal; a proof of this can be found in [27, 14.13]. Another theorem found in

Matsumura [27, 14.10] states that for a system of parameters $\mathfrak{q} = (x_1, \ldots x_d)$, $l(R/\mathfrak{q}) \geq e(\mathfrak{q})$. We say a local ring $(R, \mathfrak{m})$ is **regular** if the maximal ideal is generated by a system of parameters.

Definition 1.4 *A local commutative Noetherian ring R is an* ***isolated singularity*** *if for every non-maximal prime P we have R_P is a regular local ring.*

Let R be a local commutative Noetherian ring. We say $x_1, \ldots, x_r$ is an R-**sequence** if x_i is a nonzero divisior on $R/(x_1, \ldots, x_{i-1})$ for all $1 \leq i \leq r$. In a local Noetherian ring $(R, \mathfrak{m})$, a maximal R-sequence is the longest R-sequence $x_1, \ldots, x_r$ such that for every $x \in R$ such that $x \notin (x_1, \ldots, x_r)$ then x is a zero divisor on $R/(x_1, \ldots, x_r)$. We define the depth(R) equal to the length of the maximal R-sequence. If depth$(R) = \dim(R)$ we say the ring is **Cohen-Macaulay**. In any ring R we say that R is Cohen-Macaulay if for every maximal ideal $\mathfrak{m}$ of R, the ring $R_{\mathfrak{m}}$ is Cohen-Macaulay.

We say an ideal I is **irreducible** if we cannot write $I = J_1 \cap J_2$ where $I \subsetneq J_1$ and $I \subsetneq J_2$. A local ring $(R, \mathfrak{m})$ is **Gorenstein** if R is Cohen-Macaulay and every ideal generated by a system of parameters in R is irreducible.

1.2 Tight Closure

About ten years ago tight closure was created by Hochster and Huneke. Since its invention, tight closure has been used to attack many problems in commutative algebra including: the "homological conjectures", Big Cohen-Macaulay Modules, singularity theory, the Briançon-Skoda Theorem and more. To give a brief idea of the subject in a recent Research Report [1], Bruns referred to tight closure as "a synonym for characteristic p methods in commutative algebra." Thus far no tight closure theory is known for rings of mixed characteristic; hence, we require all our rings to contain a field. Although an equicharacteristic notion in general, tight closure is much simpler to define and understand in characteristic $p > 0$ since the notion is directly linked to the Frobenius map, which is defined as follows:

Definition 1.5 *Let R be a commutative Noetherian ring of characteristic p. The **Frobenius map**, $F : R \to R$ is defined by $F(r) = r^p$ for all $r \in R$. Similarly, we define $F^e : R \to R$ by $F(r) = r^{p^e}$.*

Note that F is an endomorphism of R since $F(a+b) = a^p + b^p$. We will pay tribute to the Frobenius in many subsequent tight closure constructs. Throughout, we will denote R viewed as an R-module after e applications of the Frobenius as $F^e(R)$.

Let R be a commutative Noetherian ring of prime characteristic $p > 0$ and I be an ideal of R. Denote varying powers of p by q. We define R^o to be the complement of the union of minimal primes.

Definition 1.6 *We say $x \in R$ is in the **tight closure** of I if there exists a $c \in R^o$ such that $cx^q \in I^{[q]}$ for all large q, where $I^{[q]}$ is generated by all qth powers of I. We denote the tight closure of I by I^*.*

Upon seeing the definition, one might wonder many things including:

1) If $I \subseteq J$ are ideals in R is $I^* \subseteq J^*$?

2) In what type of rings does the property $cx^q \in I^{[q]}$ hold for all $x \in I^*$ and all q?

3) In what kind of rings are all of the ideals tightly closed?

4) Does there exist a $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $x \in I^*$, all q and all ideals $I \subseteq R$?

The following theorem illustrates some of the basic properties of tight closure along with answering some of the above questions.

Theorem 1.7 *[14] Let R be an Noetherian ring of characteristic p and I an ideal in R.*

a) $(I^*)^* = I^*$ *and if* $I \subseteq J$ *then* $I^* \subseteq J^*$.

b) *If* R *is reduced or if* I *has positive height, then* $x \in R$ *is in* I^* *if and only if there exists a* $c \in R^o$ *such that* $cx^q \in I^{[q]}$ *for all* q.

c) *An element* $x \in R$ *is in* I^* *if and only if the image of* x *in* R/P *is in the tight closure of* $(I+P)/P$ *for every minimal prime* P *of* R.

d) $I^* \subseteq \overline{I}$, *the integral closure of* I.

e) *Let* R *be a regular local ring. Then* $I^* = I$ *for every ideal* $I \subseteq R$.

f) *If* I *is tightly closed, then* $(I : J) = \{x | xJ \subseteq I\}$ *is tightly closed for every ideal* $J \subseteq R$.

In locally excellent Noetherian rings R which are of equicharacteristic 0, Hochster defines the notion of tight closure [18, 3.1, Appendix 1] as follows. Let $I \subseteq R$ be an ideal. We say that $u \in I$ is in the tight closure of R if there exists a finitely generated $\mathbb{Z}$-subalgebra $R_{\mathbb{Z}}$ of R containing u such that, with $I_{\mathbb{Z}} = I \cap R_{\mathbb{Z}}$, one has that, for all but at most fintely many closed fibers R_κ of $\mathbb{Z} \to R_{\mathbb{Z}}$, the image u_κ of u in R_κ is in the characteristic p tight closure (as defined above) of the image of I_κ in R_κ. With such a definition for tight closure in equicharacteristic 0 the above theorem excluding (b) holds and can be found in [18, 4.1, 6.3, 6.1, Appendix 1].

The tight closure of a submodule of an R-module M is also a well defined notion.

Definition 1.8 *Let R be a commutative Noetherian ring, M be an R-module and $N \subseteq M$ a submodule of M. Define $N_M^{[q]} = ker(F^e(M) \to F^e(M/N))$. We say $x \in M$ is in the* **tight closure** *of N if there exists a $c \in R^o$ such that $cx^q \in N_M^{[q]}$ for all large q. We denote the tight closure of N in M by N_M^*.*

When M is a non-finitely generated R-module, the tight closure of submodules of M can be difficult to compute so Hochster and Huneke have defined the following:

Definition 1.9 *[14] Let R be a commutative Noetherian ring, M be an R-module and $N \subseteq M$ a submodule of M. The* **finitistic tight closure** *of N in M is*

$$N_M^{*fg} = \bigcup_{M'} (N \cap M')_{M'}^*$$

where M' ranges over all finitely generated R-modules of M.

The definition clearly indicates that ${N_M^*}^{fg} \subset N_M^*$ but this containment may be strict in general.

Property e) only partially answers 3) above. In general, there are many more rings that have the property that every ideal in the ring is tightly closed.

Definition 1.10 *A Noetherian ring R of characteristic p is called* ***weakly F-regular*** *if every ideal is tightly closed. If, in addition, R_W is weakly F-regular for every multiplicatively closed set W then we say R is* ***F-regular****.*

The following theorem contributes many examples of rings which are weakly F-regular but not necessarily regular.

Theorem 1.11 *[18] Let $(R, \mathfrak{m})$ be a local Gorenstein ring of characteristic p. Then R is weakly F-regular if and only if any parameter ideal is tightly closed.*

In fact, R is F-regular if and only if any parameter ideal is tightly closed. As not to mistake this property with F-regularity, Fedder and Watanabe [6] have given a name to rings with all parameter ideals tightly closed:

Definition 1.12 *A Noetherian ring R of characteristic p is called* ***F-rational*** *if every ideal generated by part of a system of parameters is tightly closed.*

Combining the above definition with the theorem, we see that the notions of F-regularity and F-rationality are equivalent in Gorenstein rings.

If R is equicharacteristic 0, we say R is of **F-rational type** if there exists a finitely generated $\mathbb{Z}$-subalgebra $R_{\mathbb{Z}}$ of R, one has that for all but at most fintely many closed fibers R_κ of $\mathbb{Z} \to R_{\mathbb{Z}}$, R_κ is F-rational. Work of Smith [28] and Hara [11] have shown that rings of characteristic 0 are of F-rational type if and only if they have rational singularities. Recall a normal local ring R which is essentially of finite type over a field k of characteristic 0 has rational singularities if $R^j f_*(\mathcal{O}_Z) = 0$ for all $j > 0$ where $f : Z \to Spec(R)$ a resolution of singularities.

Another important property of tight closure is called **colon capturing** where colon refers to the colon ideal $(I : J) = \{x \in R | xJ \subseteq I\}$ which is I illustrated in the following theorem:

Theorem 1.13 *[18] Let $(R, \mathfrak{m})$ be a local equidimensional ring of characteristic p which is a homomorphic image of a Cohen-Macaulay ring. Let $x_1, \ldots, x_t$ be parameters in R. Then*

$$((x_1, \ldots, x_{t-1}) :_R x_t) \subseteq (x_1, \ldots, x_{t-1})^*.$$

Question 4) above thus far has been left unanswered but it is in fact the most important question with regards to this thesis. The following definition names such elements but says nothing about their importance or even if such elements exist.

Definition 1.14 *Let R be a Noetherian ring of characteristic p. If for $c \in R^o$, $cx^q \subseteq I^{[q]}$ for all $x \in I^*$, all ideals $I \subseteq R$ and all $q = p^e$ then we say that c is a* ***test element****. The ideal generated by the test elements is called the* ***test ideal*** *and we denote it τ or τ_R.*

Recall for any ring R and R-modules $N \subseteq M$ that M is an **essential extention** of N if for any nonzero submodule $U \subseteq M$, $U \cap N \neq 0$. An injective R-module E that is an essential extention of R is called the **injective hull of** R. Using the injective hull, E, of R, we can show that the test ideal is equal to $\mathrm{Ann}_R(0_E^{*f.g.})$.

Theorem 1.15 *[14] Let R be a Noetherian ring of characteristic p. The test ideal τ is equal to $Ann_R(0_E^{*f.g.})$.*

For example, if R is regular then the test ideal is in fact R. As long as we put some finiteness assumptions on R and we assume that R is reduced then if there exist a nonzero element c such that R_c is regular then test elements exist. For our purposes we need only assume that R is F-finite.

Definition 1.16 *Let R be a Noetherian ring of characteristic p. We say that R is **F-finite** if $F(R)$ is a finite R-module.*

Theorem 1.17 *[15, Theorem 3.4] Let R be an F-finite reduced ring of characteristic p. Let c be a nonzero element of R such that R_c is regular. Then c has a power which is a test element.*

We say tight closure **persists** from R if for $\phi : R \to S$ a homomorphism of Noetherian rings of characteristic p and I an ideal of R if $x \in I^*$ implies $\phi(x)$ is in the tight closure of IS. Until the existence of test elements was known, there was no proof of persistence in any ring R.

Theorem 1.18 *[16] Let R be an F-finite reduced Noetherian ring of characteristic p and $\phi : R \to S$ a homomorphism. If $I \subseteq R$ is an ideal and $x \in I^*$ then $\phi(x)$ is in the tight closure of IS.*

There is also a notion of a parameter test element and a parameter test ideal.

Definition 1.19 *Let R be a Noetherian ring of characteristic p. An element $c \in R^o$ is a* ***parameter test element*** *if $cx^q \subseteq I^{[q]}$ for all $x \in I^*$, all parameter ideals $I \subseteq R$ and all $q = p^e$. The ideal generated by the parameter test elements is called the* ***parameter test ideal*** *and we denote it τ_{par}.*

As F-regularity and F-rationality are equal in Gorenstein local rings so too are the test ideal and the parameter test ideal. When working with parameter ideals in d-dimensional local Cohen-Macaulay rings, an important tool is the d-**the local cohomology module** with respect to the maximal ideal, $H^d_{\mathfrak{m}}(R)$. One way to define $H^d_{\mathfrak{m}}(R)$ is as follows:

Definition 1.20 *Let $(R, \mathfrak{m})$ be a d-dimensional local Cohen-Macaulay ring and $x_1, \dots, x_d$ is a system of parameters in R. Define the* ***d-th local Cohomology module of R with respect to the maximal ideal,*** $H^d_{\mathfrak{m}}(R) = \varinjlim R/(x_1^t, \dots, x_d^t)$.

In [29] Karen Smith defined the notion of F-stability in submodules of $H^d_{\mathfrak{m}}(R)$: and further defined ideals in R with F-stable annihilator in $H^d_{\mathfrak{m}}(R)$ to be F-ideals. The definition follows.

Definition 1.21 *A submodule M of $H^d_{\mathfrak{m}}(R)$ is said to be **F-stable** if $F(M) \subseteq M$. An ideal J of a local ring $(R, \mathfrak{m})$ of characteristic $p \neq 0$ is an **F-ideal** if $Ann_{H^d_{\mathfrak{m}}(R)} J$ is an F-stable module of $H^d_{\mathfrak{m}}(R)$.*

Karen Smith proves in [28] that the test ideal in a Gorenstein ring is an F-ideal. Many of our results on the test ideal depend highly on the notion of F-stability.

F-purity is a concept that is very much part of tight closure theory, but was actually studied by Fedder, Goto and Watanabe before tight closure theory came about. We define it as follows:

Definition 1.22 *A monomorphism $f : R \to S$ is **pure** if $f \otimes 1_M : R \otimes M \to S \otimes M$ is injective for all R-modules M. If $R \to F(R)$ is pure, then we say R is **F-pure**.*

It follows from the definition that if R is F-pure then if $x^p \in I^{[p]}$ then $x \in I$ for any ideal I. One might hope that F-purity and F-regularity may be the same; however, there are examples of F-pure rings that are not F-regular. There is also a parameter version of purity which we call F-injective and it is defined as follows.

Definition 1.23 *A Cohen Macaulay ring R is **F-injective** if $x^p \in I^{[p]}$ implies $x \in I$ for parameter ideals I.*

CHAPTER 2

Test ideals in reduced quotients of F-finite regular local rings

2.1 F-finite "Criteria"

In this section, $(S, \mathfrak{m})$ will always be an F-finite regular local ring of characteristic p. Set $R = S/I$. Kunz has shown in [22] that $F^e(S)$ is a faithfully flat algebra for regular local rings. Since we assumed that $F(S)$ is a finite S-module, $F^e(S)$ is also a finite S-module and hence it is free by [27, 7.10]. Fedder used the freeness of $F^e(S)$ to prove an F-purity criterion for quotients of regular local rings. We exploit this freeness in a different way to give a new proof of his criterion and to prove our main result about test ideals in F-finite reduced rings.

Lemma 2.1 *Let S be a Noetherian ring and M a free S-module. Then $\bigcap_{i \in I} I_i M = (\bigcap_{i \in I} I_i)M$ where $\{I_i\}_{i \in I}$ are ideals in S. In particular, if S is an F-finite regular local ring of characteristic p and $\{I_i\}$ a collection of ideals in S then*

$$(\bigcap I_i)^{[q]} = (\bigcap I_i)F^e(S) \bigcap S = (\bigcap I_i F^e(S)) \bigcap S = (\bigcap I_i^{[q]})$$

where $q = p^e$.

Proof. Let $\{x_j\}_{j\in J}$ be a basis for M over S. Then $M = \bigoplus_{j\in J} Sx_j$. We claim that for any ideal $\mathfrak{A} \subseteq S$ then $\mathfrak{A}M = \bigoplus_{j\in J} \mathfrak{A}x_j$. Thus we can consider

$$(\bigcap_{i\in I} I_i)M = \bigoplus_{j\in J}(\bigcap_{i\in I} I_i)x_j.$$

Trivially we can see that $(\bigcap_{i\in I} I_i)M \subseteq \bigcap_{i\in I}(I_iM)$. To see the other inclusion, we notice if $x \in \bigcap_{i\in I}(I_iM)$ then $x \in I_iM$ for all $i \in I$. Now $x = \Sigma_{j\in J}\, a_jx_j$ where $a_j \in I_i$ for all $i \in I$ and $j \in J$. This implies $a_j \in \bigcap_{i\in I} I_i$. Thus $x \in (\bigcap_{i\in I} I_i)M$. □

We note another Lemma before we proceed with the new proof for the above mentioned criterion.

Lemma 2.2 *Let $\{J_n\}$ be a collection of ideals in S such that $\bigcap J_n = I$. Then* $\bigcap(J_n^{[q]} : I) = (I^{[q]} : I)$.

Proof. Lemma 2.1 implies $\bigcap J_n^{[q]} = I^{[q]}$. Since $I^{[q]} \subseteq J_n^{[q]}$ for all $n \in \mathbb{N}$,

$$(I^{[q]} : I) \subseteq \bigcap (J_n^{[q]} : I).$$

To see the opposite containment we need only prove

$$\bigcap (J_n^{[q]} : I) \subseteq (I^{[q]} : I).$$

Let $x \in \bigcap(J_n^{[q]} : I)$. Then $Ix \subseteq \bigcap_{n\in\mathbb{N}} J_n^{[q]} = I^{[q]}$. In other words, $x \in (I^{[q]} : I)$. □

Besides the freeness of $F^e(S)$ over S, we wish to use the fact that $R = S/I$ is **approximately Gorenstein**, i.e., there exist ideals $\{q_n\}$ cofinal with $\mathfrak{m}$ such that R/q_n is a 0-dimensional Gorenstein ring for all $n \in \mathbb{N}$. To see that $R = S/I$ is approximately Gorenstein recall that $R = S/I$ is a reduced F-finite ring and use the following Theorem from Hochster [13, Theorem 1.6]:

Theorem 2.3 *Let $(R, \mathfrak{m})$ be an excellent local ring with $dim(R) \geq 1$. Then R is approximately Gorenstein if and only if*

a) $\mathfrak{m} \notin Ass(R)$ and

b) If $P \in Ass(R)$ and $dim R/P = 1$, then $R/P \oplus R/P$ is not embedable in R.

Now we give a new proof of Fedder's F-purity criterion:

Theorem 2.4 *[5, Proposition 1.7] Let $R = S/I$ where $(S, \mathfrak{m})$ is an F-finite regular local ring. Then R is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$.*

Proof. ($\Rightarrow$) Since R is an F-finite local ring, by a Theorem of Kunz [23] R is excellent. Since R is F-pure, it is reduced or in particular if $\dim(R) \geq 1$ and $P \in \mathrm{Ass}(R)$, R_P is a regular local ring. If $R/P \oplus R/P$ were embedable in R then R_P couldn't be Gorenstein, contradicting R_P regular. Combining these two points we see that R is approximately Gorenstein by 2.3. Thus, we have ideals q_n containing I with S/q_n a 0-dimensional Gorenstein ring and $\bigcap_{n\geq 1} q_n = I$.

Since S/q_n is a 0-dimensional a Gorenstein ring, $\mathrm{Soc}(S/q_n) \cong S/\mathfrak{m}$, i.e., $\mathrm{Soc}(S/q_n) = (q_n : \mathfrak{m})$ can be generated by one element, say $x_n \notin q_n$. We know from [2, exercise 3.2.15] that

$$(q_n : (q_n : \mathfrak{m})) = \mathfrak{m}.$$

Thus $(q_n : x_n) = \mathfrak{m}$. Since the Frobenius is faithfully flat on regular local rings,

$$(q_n^{[p]} : x_n^p) = \mathfrak{m}^{[p]}.$$

By Lemma 2.2, we know

$$\bigcap (q_n^{[p]} : I) = (I^{[p]} : I).$$

Since $(q_n^{[p]} : I)$ form a descending chain of ideals in S, if $(I^{[p]} : I) \subseteq \mathfrak{m}^{[p]}$ then for some large n,

$$(q_n^{[p]} : I) \subseteq \mathfrak{m}^{[p]} = (q_n^{[p]} : x_n^p).$$

Hence

$$(q_n^{[p]} : (q_n^{[p]} : x_n^p)) \subseteq (q_n^{[p]} : (q_n^{[p]} : I)).$$

But

$$x_n^p \in (q_n^{[p]} : \mathfrak{m}^{[p]}) = (q_n : \mathfrak{m})^{[p]}.$$

Thus

$$x_n^p \in (q_n^{[p]} : (q_n^{[p]} : I)).$$

We claim that

$$(q_n^{[p]} : (q_n^{[p]} : I)) = q_n^{[p]} + I.$$

We need only show that

$$(q_n^{[p]} : I) = (q_n^{[p]} : q_n^{[p]} + I),$$

since S/q_n is a 0-dimensional Gorenstein ring. First note that

$$q_n^{[p]} + I \subseteq (q_n^{[p]} : (q_n^{[p]} : I)).$$

Thus

$$(q_n^{[p]} : I) = (q_n^{[p]} : (q_n^{[p]} : (q_n^{[p]} : I))) \subseteq (q_n^{[p]} : q_n^{[p]} + I).$$

However, $I \subseteq q_n^{[p]} + I$ implies that

$$(q_n^{[p]} : q_n^{[p]} + I) \subseteq (q_n^{[p]} : I).$$

Thus $x_n^p \in q_n^{[p]} + I$. Since x_n was chosen not in q_n and $x_n^p \in q_n^{[p]}$, we conclude that R is not F-pure.

($\Leftarrow$) Suppose $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$. Let J be an ideal of S containing I. If $x^p \in J^{[p]} + I$, then we conclude

$$(I^{[p]} : I)x^p \subseteq J^{[p]}.$$

Thus

$$(I^{[p]} : I) \subseteq (J : x)^{[p]}$$

and $(J : x)^{[p]}$ is not contained in $\mathfrak{m}^{[p]}$ by assumption. Thus $x \in J$ and thus R is F-pure. □

Fedder has shown the above criterion for $q = p$. It is fairly easy to show when $q = p^e$ with $e \geq 1$ that $(I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$ is equivalent to $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$. Thus we can check F-purity on a quotient of an F-finite regular local ring for any power of p.

Theorem 2.5 *Let $R = S/I$ where S is an F-finite regular local ring. Then $(I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$ if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$.*

Proof. ($\Rightarrow$) If $q = p$ there is nothing to show. Suppose $(I^{[p]} : I) \subseteq \mathfrak{m}^{[p]}$. Apply the $(e-1)$st power of the Frobenius to get

$$(I^{[p]} : I)^{[q/p]} = (I^{[q]} : I^{[q/p]}) \subseteq \mathfrak{m}^{[q]}.$$

But

$$(I^{[q]} : I) \subseteq (I^{[q]} : I^{[q/p]}) \subseteq \mathfrak{m}^{[q]}$$

which is a contradiction.

($\Leftarrow$) Suppose there exists a q such that

$$(I^{[q]} : I) \subseteq \mathfrak{m}^{[q]}.$$

Since S is regular, $\mathfrak{m} = (x_1, \ldots, x_d)$ where $x_1, \ldots, x_d$ is a system of parameters. The socle of $S/\mathfrak{m}^{[p]}$ is

$$(\mathfrak{m}^{[p]} :_{\mathfrak{m}^{[p]}} \mathfrak{m}) = (\mathfrak{m}^{[p]}, (x_1 \ldots x_d)^{p-1})/\mathfrak{m}^{[p]}$$

If $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$ then there exists $x \in (I^{[p]} : I)$ such that x generates the socle of $S/\mathfrak{m}^{[p]}$. We conclude that

$$x = (x_1 \ldots x_d)^{p-1} + a_1 x_1^p + \ldots a_d x_d^p.$$

Note that

$$\Pi_{i=0}^{n-1} x^{p^i} \in \Pi_{i=0}^{n-1} (I^{[p]} : I)^{[p^i]} = \Pi_{i=0}^{n-1} (I^{[p^{i+1}]} : I^{[p^i]}) \subseteq (I^{[p^n]} : I).$$

By an easy computation we see that

$$\Pi_{i=0}^{n-1} x^{p^i} = x^{\frac{p^n-1}{p-1}} = (x_1 \ldots x_d)^{p^n-1} + b_1 x_1^{p^n} + \ldots + b_d x_d^{p^n}$$

for some $b_i \in R$. But this is in the socle of $S/\mathfrak{m}^{[p^n]}$. Thus $(I^{[p^n]} : I) \not\subseteq \mathfrak{m}^{[p^n]}$. □

2.2 Test ideals in F-finite reduced rings

F-pure rings are a special class of reduced rings in which $x^q \in I^{[q]}$ only if $x \in I$ for all q. Fedder and Watanabe have shown that in F-pure rings the test ideal τ is radical which implies that R/τ is reduced. Many examples including the following seemed to indicate that R/τ was in fact F-pure.

Example 2.6 *Let $R = k[[x, y, z]]/(x^2 - (y^3 + yz^4 + z^6))$ where k is an algebraically closed field of characteristic $p > 5$. By Theorem 2.4, R is F-pure since $(x^2 - (y^3 + yz^4 + z^6))^{p-1} \notin (x, y, z)^{[p]}$. We will see later in 3.34 that R has test ideal equal to the maximal ideal (x, y, z). Clearly $R/(x, y, z)$ is F-regular and hence F-pure.*

The following Theorem of of Goto and Watanabe classifies one dimensional F-pure rings containing an algebraically closed field in [10].

Theorem 2.7 *[10, Theorem 1.1] Let $(R, \mathfrak{m}, k)$ be a one dimensional local ring with k algebraically closed with positive characteristic. If R is F-finite then R is F-pure if and only if $\hat{R}$ is isomorphic to $k[[x_1, \dots, x_r]]/(x_ix_j)_{i<j}$ where r is the number of associated primes of $\hat{R}$.*

This Theorem prompts us to examine if

$$R = k[[x_1, \dots, x_d]]/I \text{ (or } R = k[x_1, \dots, x_d]/I)$$

where I is generated by squarefree monomials is in general F-pure. We observe that if I is generated by squarefree monomials $(x_1 \cdots x_d)^{p-1} \in (I^{[p]} : I)$ thus Fedder's

Criterion 2.4 implies R is F-pure. One of our goals in this section is to show that the test ideal τ in such R is generated by squarefree monomials and thus R/τ is F-pure.

To show that R/τ is F-pure when $R = S/I$ is F-pure where $(S, \mathfrak{m})$ a regular local ring, we need to show that $(f^{-1}(\tau_R)^{[p]} : f^{-1}(\tau_R)) \not\subseteq \mathfrak{m}^{[p]}$ where the canonical surjection is given by $f : S \to R$. This will be trivial if one can demonstrate that $(I^{[p]} : I) \subseteq (f^{-1}(\tau_R)^{[p]} : f^{-1}(\tau_R))$. Using some of the same techniques from the previous section, we show the following:

Theorem 2.8 *Let S be a F-finite regular local ring, $R = S/I$. If $f : S \to R$ is the canonical surjection, then $(I^{[p]} : I) \subseteq (f^{-1}(\tau_R)^{[p]} : f^{-1}(\tau_R))$.*

Proof. If $J \subseteq S$ is an ideal with $I \subseteq J$, denote $J/I \subseteq R$ as $\overline{J}$. Notice that for all $\overline{J} \subseteq R$, we have $\tau_R(\overline{J}^*)^{[p]} \subseteq \overline{J}^{[p]}$. Choose c and x in S such that $\overline{c} \in \tau_R$ and $\overline{x} \in \overline{J}^*$. We note that $\overline{cx^p} \in \overline{J}^{[p]}$. In other words after pulling elements and ideals back to S, we have $cx^p \in J^{[p]} + I$. Thus

$$f^{-1}(\tau_R)(f^{-1}(\overline{J}^*))^{[p]} \subseteq J^{[p]} + I.$$

Let $w \in (I^{[p]} : I)$, so that

$$wf^{-1}(\tau_R)(f^{-1}(\overline{J}^*))^{[p]} \subseteq J^{[p]} + I^{[p]} \subseteq J^{[p]}$$

in S. Thus

$$wf^{-1}(\tau_R) \subseteq (J^{[p]} :_S (f^{-1}(\overline{J}^*))^{[p]}) = (J :_S f^{-1}(\overline{J}^*))^{[p]}.$$

So

$$wf^{-1}(\tau_R) \subseteq \bigcap_{I \subseteq J} (J :_S f^{-1}(J^*))^{[p]}.$$

Using Lemma 2.1, we see that

$$\bigcap_{I \subseteq J} (J :_S f^{-1}(\overline{J}^*))^{[p]} = (\bigcap_{I \subseteq J} (J :_S f^{-1}(\overline{J}^*)))^{[p]}.$$

If

$$\bigcap_{I \subseteq J} (J :_S f^{-1}(\overline{J}^*)) = f^{-1}(\tau_R)$$

we are done. To see this we note that

$$v \in \bigcap_{I \subseteq J} (J :_S f^{-1}(\overline{J}^*))$$

if and only if

$$v \in (J :_S f^{-1}(\overline{J}^*)) \text{ for all } \overline{J} \subseteq R$$

if and only if

$$vf^{-1}(\overline{J}^*) \subseteq J \text{ for all } \overline{J} \subseteq R$$

if and only if

$$f(v)\overline{J}^* \subseteq \overline{J} \text{ for all } \overline{J} \subseteq R$$

if and only if

$$f(v) \in (\overline{J} :_R \overline{J}^*) \text{ for all } \overline{J} \subseteq R$$

if and only if

$$f(v) \in \tau_R$$

if and only if

$$v \in f^{-1}(\tau_R).$$

Thus $w \in (f^{-1}(\tau_R)^{[p]} : f^{-1}(\tau_R))$. □

Notice that the proof holds for any $q = p^e$. Fedder has noted that to discuss F-purity of a ring R, we must assume that R is reduced. The proof of the next Theorem relies on the fact that in an F-finite, F-pure ring the test ideal is nonzero and has positive height. To see the test ideal is nonzero recall the following Theorem of Hochster and Huneke:

Theorem 2.9 *[15, Theorem 3.4] Let R be an F-finite reduced ring of characteristic p. Let c be any nonzero element of R such that R_c is regular. Then some power of c is a test element.*

Now we have the key ingredients for the main theorem of this section.

Theorem 2.10 *Let $R = S/I$ be an F-finite, F-pure ring. Let τ_R be the pullback of the test ideal of R in S, then S/τ_R is F-pure and $ht(\tau) \geq 1$.*

Proof. Recall that F-pure rings are reduced. For each minimal prime P/I in R, we know that $R_{P/I} = (S/I)_{P/I} = S_P/P$ which is a field. The regular locus is open in an excellent Noetherian local ring. A Theorem of Kunz [23] shows that F-finite rings are excellent. We conclude that the regular locus of R is nonempty. Thus if we choose c to be an element such that the primes not containing c are contained inside the regular locus then R_c is regular and thus some power is in the test ideal by Theorem 2.9. Since R is reduced, c is not nilpotent. It follows that $\mathrm{ht}((c)) = 1$. Therefore, $\mathrm{ht}(\tau_R) \geq 1$.

Define $R_1 = R/\tau_R \cong S/f^{-1}(\tau_R)$. Since R is F-pure, Fedder's F-purity criterion, Theorem 2.4, implies that $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$. Applying Theorem 2.8 we see that $(f^{-1}(\tau_R)^{[p]} : f^{-1}(\tau_R)) \not\subseteq \mathfrak{m}^{[p]}$. Thus another application of Theorem 2.4 implies that R_1 is F-pure. □

An immediate Corollary of Theorem 2.10 is the following filtration of R by test ideals. In [30] Smith shows the test ideal is a D-modules where D is the ring of differential operators on R; thus, the following filtration is also an example of a filtration of R by D-modules.

Corollary 2.11 *Let $R = S/I$ be an F-finite, F-pure ring. Then there exists a unique filtration of R, $\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_{c-1} \subseteq \tau_c$ such that τ_{i+1}/τ_i is the test ideal for R/τ_i and R/τ_i are all F-pure for all i.*

Proof. Applying 2.10 to $R_i = S/\tau_{i-1}$ where τ_{i-1} is the pullback of the test ideal of R_{i-1} in S we see that R_i is F-pure for all i and we get the unique filtration $0 \subseteq \tau_R \subseteq \tau_1/I \ldots \subseteq \tau_i/I \subseteq \ldots \tau_c$ where is the last such i such that $\tau_c \neq R$ and in which all of the R_i are F-pure and $\mathrm{ht}(\tau_{R_{i+1}}/\tau_{R_i}) \geq 1$. □

In fact, Theorem 2.10 implies the following Theorem of Fedder and Watanabe.

Theorem 2.12 *[6, Proposition 2.5] Let R be an F-pure ring. Then the test ideal is radical.*

As noted above, we will find an algorithm to compute the test ideal of reduced rings R where $R = T/I$, $T = k[x_1, \ldots, x_n]$ or $T = k[[x_1, \ldots, x_n]]$ and I is an ideal generated by square-free monomials in $x_1, \ldots, x_n$. For a thorough discussion of monomials see [4, 15.1] and for a reference to monomials in the tight closure context refer to [14, 7.3]. For the following theorem we note that the sum of two ideals generated by monomials is an ideal generated by monomials and the intersection of two such ideals is also generated by monomials.

Theorem 2.13 *Let $R = T/I$ with T as above where $I = P_1 \cap \ldots \cap P_n$ is generated by monomials and R/P_i are regular. Set $J = \Sigma(P_1 \cap \ldots \cap \hat{P}_i \cap \ldots \cap P_n)$. Then $J = \tau = \tau_{par}$.*

Proof. To show that $J \subseteq \tau$ we need only see that

$$(P_1 \cap \ldots \cap \hat{P}_i \cap \ldots \cap P_n)I^* \subseteq I.$$

However, since $I^* = \bigcap(I + P_i)$,

$$(P_1 \cap \ldots \cap \hat{P}_i \cap \ldots \cap P_n)I^* \subseteq (P_1 \cap \ldots \cap \hat{P}_i \cap \ldots \cap P_n)(I + P_i)$$

$$= (P_1 \cap \ldots \cap \hat{P}_i \cap \ldots \cap P_n)I \subseteq I.$$

To show the other inclusion, take a nonzero

$$x_i \in (P_1 \cap \ldots \cap \hat{P}_i \cap \ldots \cap P_n).$$

Then $x_1 + \ldots + x_n$ is not in the union of minimal primes and it is the generator of a parameter ideal. Note that $x_i \in (x_1 + \ldots + x_n)^*$ since

$$(x_1 + \ldots + x_n)x_i^q = (x_1 + \ldots + x_n)^q x_i$$

for all i. Take a minimal prime $\mathfrak{p}$ over J. If $\tau \not\subseteq J$ then by a prime avoidance argument there exists a $c \in \tau$ such that $c \notin \mathfrak{p}$. Thus c is a unit in $R_\mathfrak{p}$ and

$$(x_1 + \ldots + x_n)^* R_\mathfrak{p} = (x_1 + \ldots + x_n) R_\mathfrak{p}.$$

In $R_\mathfrak{p}$, we have

$$x_i \in (x_1 + \ldots + x_n) R_\mathfrak{p}$$

i.e., $x_i = a(x_1 + \ldots + x_n)$. We see

$$(1 - a)x_i = a(x_1 + \ldots + \hat{x}_i + \ldots + x_n).$$

Since $R_\mathfrak{p}$ is local, either a or $1 - a$ is a unit. Thus either

$$x_i \in (x_1 + \ldots + \hat{x}_i + \ldots + x_n)$$

which implies $x_i \in P_i$ or

$$(x_1 + \ldots + \hat{x}_i + \ldots + x_n) \in (x_i)$$

which implies $(x_1 + \ldots + \hat{x}_i + \ldots + x_n) \in (P_1 \bigcap \ldots \bigcap \hat{P}_i \bigcap \ldots \bigcap P_n)$. From both of these conclusions, we can conclude $x_i = 0$ for some i. But we assumed the x_i were nonzero.

Note this argument also shows that $J = \tau_{par}$, since we can assume that the ideal I in the first sentence of the proof is a parameter ideal and replace τ by τ_{par} throughout the rest of the proof. □

Recall the strong test ideal from [20]:

Definition 2.14 *Let R be a commutative Noetherian ring. A strong test ideal is an ideal $J \subseteq \tau$ such that $JI^* = JI$ for all ideals I in R. We denote $J = \tau_{str}$ as the largest one.*

Corollary 2.15 *If all the assumptions of Theorem 2.13 hold, then $J = \tau_{str}$.*

Proof. We know that $\tau_{str} \subseteq \tau = J$. And

$$(P_1 \cap \ldots \cap \hat{P}_i \cap \ldots \cap P_n)I^* \subseteq (P_1 \cap \ldots \cap \hat{P}_i \cap \ldots \cap P_n)(I + P_i)$$

$$= (P_1 \cap \ldots \cap \hat{P}_i \cap \ldots \cap P_n)I$$

implies that $JI^* = JI$. Thus $\tau_{str} = J$. □

Note that we can also prove Theorem 2.13 as a corollary using the following more general Theorem.

Theorem 2.16 *Let R be an F-pure ring and I and J be ideals in R such that $I \cap J = (0)$. Suppose also that $Ann(I) = J$ and $Ann(J) = I$. Let $\tau_{R/I}$ and $\tau_{R/J}$ be test ideals in R/I and R/J lifted back to R respectively. If $\mathfrak{a} = I \cap \tau_{R/J} + J \cap \tau_{R/I}$ is a radical ideal then $\tau = \mathfrak{a}$.*

Proof. We need only see that $\mathfrak{a} \subseteq \tau$, since the other containment follows from a proof similar to showing that $\tau \subseteq J$ in Theorem 2.13. It is enough to show that $I\tau_{R/J} \subseteq \tau$ and $J\tau_{R/I} \subseteq \tau$, since

$$\begin{aligned} I\tau_{R/J} + J\tau_{R/I} &\subseteq I \bigcap \tau_{R/J} + J \bigcap \tau_{R/I} \\ &\subseteq \text{Rad}(I \bigcap \tau_{R/J}) + \text{Rad}(J \bigcap \tau_{R/I}) \\ &\subseteq \text{Rad}(I\tau_{R/J}) + \text{Rad}(J\tau_{R/I}) \\ &\subseteq \text{Rad}(I\tau_{R/J} + J\tau_{R/I}) \\ &\subseteq \mathfrak{a}. \end{aligned}$$

If R is F-pure then 2.12 shows τ is radical. Let $\mathfrak{b}$ be an ideal in R,

$$\mathfrak{b}^* = f^{-1}((\mathfrak{b} + I)^*) \bigcap g^{-1}((\mathfrak{b} + J)^*)$$

where $f : R \to R/I$ and $g : R \to R/J$ are the natural surjections. Note that

$$(I\tau_{R/J})\mathfrak{b}^* \subseteq I(\mathfrak{b} + J) \subseteq I\mathfrak{b} \subseteq \mathfrak{b}.$$

By symmetry

$$J\tau_{R/I}\mathfrak{b}^* \subseteq \mathfrak{b}.$$

□

To give some examples of F-pure filtrations, we use the above Theorems to prove the following:

Theorem 2.17 *Let* $T = k[[x_1, \ldots, x_n]]$ *or* $T = k[x_1, \ldots, x_n]$. *Set* $R = T/I$ *where*

$$I = < x_{i_1} \ldots x_{i_d} >_{1 \le i_1 < \ldots < i_d \le n} .$$

Then $\tau = < x_{i_1} \ldots \hat{x_{i_r}} \ldots x_{i_d} >_{1 \le i_1 < \ldots < i_d \le n}$.

Proof. Note that I is generated by $\binom{n}{d}$ elements each missing $(n-d)$ of the x_i's. If $\{x_{j_1}, \ldots, x_{j_{n-d+1}}\}$ are $(n-d+1)$ distinct x_i's, then for a fixed $\{x_{i_1}, \ldots, x_{i_d}\}$ some x_{j_n} is a factor of $x_{i_1} \ldots x_{i_d}$ (i.e., $j_n = i_m$ where $x_{i_m} \in \{x_{i_1}, \ldots, x_{i_d}\}$) since $x_{i_1} \ldots x_{i_d}$ is only missing $(n-d)$ of the x_i's. Take $P_{j_1 \ldots j_{n-d+1}} = (x_{j_1}, \ldots, x_{j_{n-d+1}})$. The $P_{j_1 \ldots j_{n-d+1}}$ are minimal primes and there are $\binom{n}{n-d+1}$ of them. Theorem 2.13 says that $\tau = \Sigma \bigcap_{j_i \neq k_i} P_{j_1 \ldots j_{n-d+1}}$ some $\{k_1, \ldots, k_{n-d+1}\}$. But

$$\bigcap_{j_i \neq k_i} P_{j_1 \ldots j_{n-d+1}} = (x_{l_1} \ldots x_{l_{d-1}}, < x_{k_r} x_{l_1} \ldots x_{l_{d-1}} >)$$

where $\{l_1, \ldots l_{d-1}\}$ is the complement of $\{k_1, \ldots, k_{n-d+1}\}$ in $\{1, \ldots n\}$. Thus

$$\tau = < x_{l_1} \ldots x_{l_{d-1}} >_{1 \leq l_1 < \ldots < l_{d-1} \leq n} = < x_{i_1} \ldots \hat{x_{i_r}} \ldots x_{i_d} >_{1 \leq i_1 < \ldots < i_d \leq n} .$$

□

Example 2.18 *1. Let $R = k[[x,y,z]]/(xy, xz, yz)$ where k is algebraically closed. By Theorem 2.7, the ring R is F-pure. By Theorem 2.11 then R/τ is also F-pure. Thus we need to compute τ. By Theorem 2.17, we have $\tau = (x,y,z) = \mathfrak{m}$. So the filtration is $(xy, xz, yz) \subseteq \mathfrak{m} \subseteq R$.*

2. Let $R = k[[x,y,z,w]]/(xyz, xyw, xzw, yzw)$ where k is algebraically closed. Using Fedder's F-purity criterion, the ring R is F-pure. By Theorem 2.11, we also know that $R/\tau = R_1$ is F-pure. By Theorem 2.17, we have $\tau = (xy, xz, xw, yz, yw, zw)$. As in the previous example, $\tau_{R_1} = \mathfrak{m}$. Thus the F-pure filtration is

$$(xyz, xyw, xzw, yzw) \subseteq (xy, xz, xw, yz, yw, zw) \subseteq \mathfrak{m} \subseteq k[[x,y,z,w]].$$

3. Let $R = k[[x,y,z]]/(xy,yz)$ where k is algebraically closed. By Fedder's F-purity criterion, R is F-pure. Note that $P_1 = (y)$ and $P_2 = (x,z)$. By Theorem 2.13, we have $\tau = (x,y,z)$. So the filtration is $(xy,xz,yz) \subseteq \mathfrak{m} \subseteq R$.

4. Let $R = k[[x,y,z,w]]/(xy,zw)$ where k is algebraically closed. Using Fedder's F-purity criterion, R is F-pure. By Theorem 2.11, we also know that $R/\tau = R_1$ is F-pure. By Theorem 2.17, we have $\tau = (xy,xz,xw,yz,yw,zw)$. As in the previous example, $\tau_{R_1} = \mathfrak{m}$. Thus the F-pure filtration is

$$(xy,zw) \subseteq (xy,xz,xw,yz,yw,zw) \subseteq \mathfrak{m} \subseteq k[[x,y,z,w]].$$

One hopes the above filtration is a partial classifying tool for n-dimensional F-pure rings containing an algebraically closed field.

2.3 Powers of tight closures of ideals

What power of I^* is contained in I? When the closure operation we are dealing with is integral closure or $\overline{I}$ 3.7, then in an F-pure ring we know the following Theorem of Huneke [17, Proposition 4.9]:

Theorem 2.19 *Let R be a d-dimensional Noetherian Cohen-Macaulay local ring which is F-pure. Then for all ideals I*

$$\overline{I^{d+1}} \subseteq I.$$

Theorem 2.19 can be improved slightly. The proof is similar to the proof of 2.19.

Theorem 2.20 *Let R be a d-dimensional Noetherian Cohen-Macaulay local ring which is F-pure. Then for all ideals I*

$$\overline{I^{d+n}} \subseteq I^n.$$

Proof. Without loss of generality assume that I is a minimal reduction 1.3. Thus I is generated by at most d elements. Take $u \in \overline{I^{d+n}}$. Fix an integer k such that $u^{k+m} \in I^{m(d+n)}$. Set $m = q - k$. Then $u^q \in I^{(q-k)(d+n)}$. Rewriting the power $(q-k)(d+n) = q(d+n-1) + q - k(d+n)$ and choosing $q > k(d+n)$ we see that $u^q \in I^{q(d+n-1)} \subseteq I^{[q]}I^{q(d+n-2)} \subseteq \ldots \subseteq (I^{[q]})^{n-1}I^{qd} \subseteq (I^{[q]})^n = (I^n)^{[q]}$. Since R is F-pure then $u \in I^n$. □

Both of the above Theorems imply $(I^*)^{d+1} \subseteq I$. But we would like a better bound. Theorem 2.11 gives us a hint on how to find this bound in F-pure rings. Recall the setup for Theorem 2.11. Let $R = S/I$ be an F-finite, F-pure ring. As noted earlier in F-pure local rings test elements exist. Let τ_0 be the test ideal of R. As we noted before, $\mathrm{ht}(\tau_0) \geq 1$. Since R/τ_0 is again F-pure, we can define τ_1 to be the test ideal of R/τ_0. Continue until $\tau_c = R$ for some c. Now our filtration looks like $\tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_{c-1} \subseteq \tau_c$ such that τ_{i+1}/τ_i is the test ideal for R/τ_i and R/τ_i are all F-pure and $\mathrm{ht}(\tau_{i+1}/\tau_i) \geq 1$. To make the following Theorems' proofs and statments easier we shall relabel the above filtration setting $i = c - i$. Thus our new filtration looks like: $\tau_c \subseteq \tau_{c-1} \subseteq \ldots \subseteq \tau_1 \subseteq \tau_0$ with τ_i/τ_{i+1} the test ideal for R/τ_{i+1} and $\mathrm{ht}(\tau_i/\tau_{i+1}) \geq 1$. To make the proofs of our Theorems more understandable recall Theorem 1.18:

Theorem 2.21 *[16, Theorem 6.24] Let $\phi : R \to S$ be a homomorphism of F-finite reduced Noetherian rings of characteristic p. Let I be an ideal of R and $w \in I^*$. Then $\phi(w)$ is in the tight closure of IS.*

The notion in the above theorem is referred to as persistence.

Theorem 2.22 *Let $R = S/I$, with S an F-finite regular local ring. Suppose that R is F-pure and J is an ideal of R. Let $c+1$ be the length of the unique filtration described above. Suppose for some $t \in \{0, 1, \ldots, c\}$ we have $J \subseteq \tau_i$ for $i \leq t$. Then $(J^*)^{c-t+2} \subseteq J$.*

Proof. Note that $\text{Rad}(J) = \text{Rad}(J^*)$. Thus if $J \subseteq \tau_t$ then $J^* \subseteq \tau_t$ since τ_t is reduced. We know by Theorem 1.18, that if x is in the tight closure of an ideal J of R then $x + \tau_i$ is in the tight closure of $J + \tau_i$ in R/τ_i for all i. In the ring R/τ_{t+1} we know the test ideal is τ_t/τ_{t+1}. Thus by persistence we know that

$$\tau_t J^* \subseteq J + (\tau_{t+1} \cap J^*).$$

Note that multiplying by J^* gives us the following:

$$\tau_t (J^*)^2 \subseteq J + (\tau_{t+1} \cap J^*)J^* \subseteq J + (\tau_{t+2} \cap J^*)$$

again by persistence. If we continue to multiply by J^* after $c-t+1$ steps we have

$$\tau_t (J^*)^{c-t+1} \subseteq J.$$

Thus $(J^*)^{c-t+2} \subseteq J$. □

This result can be slightly improved if τ_t is not contained in the union of the associated primes of J.

Theorem 2.23 *Let $R = S/I$, with S an F-finite regular local ring. Suppose that R is F-pure and J/I is an ideal of R. Let $c+1$ be the length of the unique filtration described above. Suppose for some $t \in \{0, 1, \ldots, c\}$ we have $J \subseteq \tau_i$ for $i \leq t$. If $\tau_t \not\subseteq Ass(J)$ then $(J^*)^{c-t+1} \subseteq J$.*

Proof. Following the proof of Theorem 2.22, we see that

$$\tau_t (J^*)^{c-t+1} \subseteq J.$$

Thus

$$(J^*)^{c-t+1} \subseteq (J : \tau_t) = J$$

since $\tau_t \not\subseteq \text{Ass}(J)$. □

We see from Theorem 2.23 that the τ_i not contained in the associated primes of an ideal J/I can virtually be ignored. In effect, for each J/I we can reduce the length of the filtration by successive test ideals from $c+1$ to $c-s$ where s is the largest integer such that $\tau_s \not\subseteq \text{Ass}(J)$. We relabel τ_s by τ_0 and for all $i > s$, τ_i becomes τ_{i-s}. Thus our new filtration looks like $\tau_{c-s-1} \subseteq \ldots \subseteq \tau_1 \subseteq \tau_0$ and it has all the same properties of the above filtration. We can reformulate the statements of both 2.22 and 2.23 in terms of our new filtration in the following two Theorems respectively:

Theorem 2.24 *Let $R = S/I$, with S an F-finite regular local ring. Suppose that R is F-pure and J/I is an ideal of R. Let $c-s$ be the length of the unique filtration described above. Suppose for some $r \in \{0, 1, \ldots, c-s-1\}$ we have $J \subseteq \tau_i$ for $i \leq r$. Then $(J^*)^{c-s-r+1} \subseteq J$.*

Theorem 2.25 *Let $R = S/I$, with S an F-finite regular local ring. Suppose that R is F-pure and J/I is an ideal of R. Let $c-s$ be the length of the unique filtration described above. If $J \subseteq \tau_0$ then $(J^*)^{c-s} \subseteq J$.*

From these new statements we can estimate a bound on the power of the tight closure of an ideal which is contained in the ideal in terms of heights of the associated primes of the ideal J. Note since τ_0 is the only element of our filtration not contained in any associated primes of J and $\mathrm{ht}(\tau_i) > \mathrm{ht}(\tau_{i+1})$ for all i, since $\mathrm{ht}(\tau_i/\tau_{i+1}) \geq 1$, we have $c - s - 1 \leq \max\{\mathrm{ht}(P_i) | P_i \subseteq \mathrm{Ass}(J)\}$. As a consequences of 2.24 and 2.25 we have:

Corollary 2.26 *Let $R = S/I$, with S an F-finite regular local ring. Suppose that R is F-pure and J/I is an ideal of R. Let $c-s$ be the length of the unique filtration described above. Suppose for some $r \in \{0, 1, \ldots, c-s-1\}$ we have $J \subseteq \tau_i$ for $i \leq r$. Then $(J^*)^{h-r+2} \subseteq J$ where $h = max\{ht(P_i) | P_i \subseteq Ass(J)\}$.*

Corollary 2.27 *Let $R = S/I$, with S an F-finite regular local ring. Suppose that R is F-pure and J/I is an ideal of R. Let $c-s$ be the length of the unique filtration described above. If $J \subseteq \tau_0$ then $(J^*)^{h+1} \subseteq J$ where $h = max\{ht(P_i) | P_i \subseteq Ass(J)\}$.*

CHAPTER 3

Test ideals in Gorenstein domains

3.1 Test Elements

Now we will concern ourselves with rings that are Cohen-Macaulay isolated singularities defined in Definition 1.4. We would like to show a few interesting facts about parameter test elements. The fact that we are in an isolated singularity guarantees that they exist [14].

Theorem 3.1 *Let R be a local d-dimensional Gorenstein ring. Then*

$$(x_1, \dots, x_d)^* = ((x_1, \dots, x_d) : \tau)$$

where $(x_1, \dots, x_d)$ is a system of parameters.

Proof. It is trivial to see that

$$(x_1, \dots, x_d)^* \subseteq ((x_1, \dots, x_d) : \tau).$$

Thus we need to show that if

$$x \in ((x_1, \dots, x_d) : \tau), \text{ then } x \in (x_1, \dots, x_d)^*.$$

But in [28, 4.5], Smith proves that τ is an F-ideal and thus by [28, 3.6]

$$\tau x \subseteq (x_1, \dots, x_d) \text{ implies } \tau x^q \subseteq (x_1^q, \dots, x_d^q)$$

for all q which are powers of p. Thus $x \in (x_1, \dots, x_d)^*$. ☐

Proposition 3.2 *Let R be a local d-dimensional Cohen-Macaulay ring. Then $\tau_{par} = (x_1, \dots, x_d) : (x_1, \dots, x_d)^*$ where $(x_1, \dots, x_d)$ is a system of parameters which are test elements.*

Proof. By [28, 4.4]

$$\tau_{par} = \{c \in R | c(x_1^t, \dots, x_d^t)^* \subseteq (x_1^t, \dots, x_d^t), \text{ for all } t \in \mathbb{N}\},$$

where $x_1, \dots, x_d$ is a fixed system of parameters. We need to show

$$((x_1, \dots, x_d) : (x_1, \dots, x_d)^*) \subseteq \tau_{par}.$$

Let $c \in ((x_1, \dots, x_d) : (x_1, \dots, x_d)^*)$. We need only check

$$c(x_1^t, \dots, x_d^t)^* \subseteq (x_1^t, \dots, x_d^t)$$

by the above. Since $x_1, \dots, x_d$ are test elements, if $z \in (x_1^t, \dots, x_d^t)^*$ then

$$z \in ((x_1^t, \dots, x_d^t) : (x_1, \dots, x_d)) = (x_1^t, \dots, x_d^t, (x_1 \cdots x_d)^{t-1}).$$

Without loss of generality, we may assume $z = u(x_1 \cdots x_d)^{t-1}$. But then

$$u(x_1 \cdots x_d)^{t-1} \in (x_1^t, \dots, x_d^t)^*.$$

Hence by colon capturing [14], $u \in (x_1, \dots, x_d)^*$ and $cu \in (x_1, \dots, x_d)$. This implies that

$$cz \in (x_1^t, \dots, x_d^t).$$

Thus by [28, 4.4], we have $c \in \tau_{par}$. ☐

Theorem 3.3 *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local isolated singularity. If $x_1, \ldots, x_d$ is a system of parameters and $x_1, \ldots, x_{d-1}$ are parameter test elements then $x_d^{t-1} \in \tau_{par}$ if and only if $(x_1, \ldots, x_d^t)^* \subseteq (x_1, \ldots, x_d)$.*

Proof. ($\Rightarrow$) If $x_d^{t-1} \in \tau_{par}$ then

$$x_d^{t-1}(x_1, \ldots, x_d^t)^* \subseteq (x_1, \ldots, x_d^t).$$

Thus

$$(x_1, \ldots, x_d^t)^* \subseteq (x_1, \ldots, x_d^t) : x_d^{t-1} = (x_1, \ldots, x_d).$$

($\Leftarrow$) Since R is an isolated singularity R_P is a regular ring for all nonmaximal primes P. Since x_d is part of a regular sequence, it is not nilpotent. Thus the primes of R_{x_d} are in one to one correspondence with the primes in R not containing x_d. The primes in R not containing x_d are not maximal and since $(R_{x_d})_{PR_{x_d}} \cong R_P$, we easily see that R_{x_d} is regular. Thus some power of x_d is a test element. If x_d^n is a test element for $n > t-1$ then, using Theorem 3.2, we get that $\tau_{par} = ((x_1, \ldots, x_d^n) : (x_1, \ldots, x_d^n)^*)$. If

$$x_d^{n-1} \notin ((x_1, \ldots, x_d^n) : (x_1, \ldots, x_d^n)^*)$$

then

$$(x_1, \ldots, x_d^n)^* \nsubseteq ((x_1, \ldots, x_d^n) : x_d^{n-1}) = (x_1, \ldots, x_d).$$

$\square$

Corollary 3.4 *Let $(R, \mathfrak{m})$ be a d-dimensional local Gorenstein isolated singularity. If $x_1, \ldots, x_d$ is a system of parameters and $x_1, \ldots, x_{d-1}$ are test elements then $x_d^{t-1} \in \tau$ if and only if $(x_1, \ldots, x_d^t)^* \subseteq (x_1, \ldots, x_d)$.*

Proof. In a Gorenstein ring, $\tau_{par} = \tau$. Thus Theorem 3.3 translates directly into $x_d^{t-1} \in \tau$ if and only if $(x_1, \ldots, x_d^t)^* \subseteq (x_1, \ldots, x_d)$. □

3.2 One-dimensional domains with test ideal the maximal ideal

It would be interesting to know Theorem 3.2 holds for Cohen-Macaulay local rings also, but it is not known that the parameter test ideal is an F-ideal in a generic Cohen-Macaulay isolated singularity. For one-dimensional domains with an infinite residue field we can prove Theorem 3.2, but the proof relies on the fact that in a one-dimensional domain with an infinite residue field, $(x)^* = \overline{(x)}$ [18], where $\overline{(x)}$ is the integral closure of the ideal (x) in R.

Let us recall some definitions about integral extensions. For more detail a good source is [27].

Definition 3.5 *Let R be a Noetherian ring. Suppose S is an R-algebra and $x \in S$. We say x is* ***integral*** *over R if x is a root of a monic polynomial $f(t) \in R[t]$. We say S is* ***integral*** *over R if for every element $x \in S$ we have x is integral over R.*

Definition 3.6 *Let R be a Noetherian domain and K its fraction field. We say $x \in K$ is in the* ***integral closure*** *of R, denoted $\overline{R}$, if x is integral over R. We say that R is* ***integrally closed*** *if $R = \overline{R}$.*

Similarly we can define what it means for elements of a Noetherian ring R to be integral over an ideal I of R.

Definition 3.7 *Let R be a Noetherian ring and $I \subseteq R$ an ideal in R. We say $x \in R$ is* ***integral*** *over I if there exists a monic polynomial*

$$f(t) = t^n + a_1 t^{n-1} + \ldots + a_n \in R[t]$$

such that $a_i \in I^i$ for all $i \in \{1, \ldots, n\}$. The set of elements integral over I form an ideal denoted $\overline{I}$. We call $\overline{I}$ the ***integral closure*** *of I in R. I is said to be* ***integrally closed*** *if $I = \overline{I}$.*

We show that in any integral domain R then $x\overline{R} \cap R = \overline{(x)}$.

Theorem 3.8 *Let R be an integral domain. Let x be an nonzero element of R. Then $x\overline{R} \cap R = \overline{(x)}$.*

Proof. First we will show that $x\overline{R} \cap R \subseteq \overline{(x)}$. Suppose $w \in x\overline{R} \cap R$. Then there exists a $y \in \overline{R}$ such that $w = yx \in R$. By definition y is the root of a monic polynomial

$$f(t) = t^n + a_1 t^{n-1} + \ldots + a_n \in R[t].$$

Set

$$g(t) = t^n + x a_1 t^{n-1} + \ldots + x^n a_n.$$

Note that

$$\begin{aligned} g(xy) &= x^n y^n + x a_1 x^{n-1} y^{n-1} + \ldots + x^n a_n \\ &= x^n (y^n + a_1 y^{n-1} + \ldots + a_n) \\ &= x^n f(y) = 0. \end{aligned}$$

Thus by definition $w = xy \in \overline{(x)}$.

Now we show that $\overline{(x)} \subseteq x\overline{R} \cap R$. Suppose $y \in \overline{(x)}$. Then y is the root of a monic polynomial of the form

$$g(t) = t^n + x a_1 t^{n-1} + \ldots + x^n a_n$$

where $a_i \in R$. Define $f(t) = x^n(t^n + a_1 t^{n-1} + \ldots + a_n)$. Note

$$\begin{aligned} f(\frac{y}{x}) &= x^n((\frac{y}{x})^n + a_1(\frac{y}{x})^{n-1} + \ldots + a_n) \\ &= y^n + x a_1 y^{n-1} + \ldots + a_n \\ &= g(y) = 0. \end{aligned}$$

Thus $\frac{y}{x} \in \overline{R}$. Since R is a domain, $y \in x\overline{R}$. But we assumed $y \in R$; therefore, $y \in x\overline{R} \cap R$. □

We can also show in one-dimensional domains that the test ideal is equal to the conductor. Recall the following definition:

Definition 3.9 *Let R be a domain. We define the* **conductor**, $\mathfrak{c}$, *to be the ideal*

$$\{c \in R | c = \phi(1), \phi \in \ Hom_R(\overline{R}, R)\}.$$

Remark 3.10 *The conductor is actually isomorphic to $Hom_R(\overline{R}, R)$.*

Proof. To see this, consider the homomorphism $\psi : \mathrm{Hom}_R(\overline{R}, R) \to \mathfrak{c}$ defined by $\psi(\phi) = \phi(1)$. It is clear that ψ is onto. We need only see that ψ is one to one. Suppose not, then $\psi(\phi) = \psi(\rho)$ i.e. $\phi(1) = \rho(1)$ But $s\phi(\frac{r}{s}) = r\phi(1)$. Thus $\phi(\frac{r}{s}) = \frac{r}{s}\phi(1)$. The same is true for ρ. Thus $\phi = \rho$. □

We can also show that the conductor, $\mathfrak{c}$ is, in fact, equal to $\bigcap_{x\in R}(x) : \overline{(x)}$ which is the key to the proof of the following.

Theorem 3.11 *Let R be a 1-dimensional local domain with infinite residue field. k. Then the parameter test ideal, τ_{par}, is the conductor.*

Proof. As noted above $(x)^* = \overline{(x)}$ in a domain. If we can show

$$\mathfrak{c} = \bigcap_{x\in R}(x) : \overline{(x)}$$

then by the definition of the parameter test ideal $\mathfrak{c} = \tau_{par}$. Let $\phi \in \mathfrak{c}$. Then

$$\phi(x\overline{R}) = x\phi(\overline{R}) \subseteq (x)$$

for all $x \in R$. By Theorem 3.8, $x\overline{R} \cap R = \overline{(x)}$. Thus $\overline{(x)}\phi(1) \subseteq (x)$ for all $x \in R$. Thus

$$\phi(1) \in \bigcap_{x\in R} (x) : \overline{(x)}$$

which gives us

$$\mathfrak{c} \subseteq \bigcap_{x\in R}(x) : \overline{(x)}.$$

The other inclusion is as follows. Suppose that

$$y \in (\bigcap_{x\in R}(x) : \overline{(x)}) \backslash \mathfrak{c}.$$

Then $yz \notin R$ for some $z \in \overline{R}$. But $z = \frac{w}{x}$ where both w and x are in R. Thus $yw \notin xR$. However, by Theorem 3.8, we have $w \in x\overline{R} \cap R = \overline{(x)}$ which gives us that $y \notin (x) : \overline{(x)}$, and that is a contradiction. □

Remark 3.12 *In a one-dimensional local domain R with infinite residue field, the parameter test ideal τ_{par} is the test ideal τ.*

Proof. Note every ideal I in a one-dimensional local domain has the same integral closure as some principal ideal $(x) \subseteq I$. Thus

$$(x)^* \subseteq I^* \subseteq \overline{I} = \overline{(x)} = (x)^*.$$

Hence $I^* = (x)^*$. Recall $\tau = \bigcap_{I \subseteq R}(I : I^*)$ and $\tau_{par} = \bigcap_{(x) \subseteq R}((x) : (x)^*)$, since every nonzero element in a one-dimensional domain is a system of parameters. Note for all ideals $I \subseteq R$, there exists some y such that $(y)^* = I^*$ and

$$((y) : (y)^*) \subseteq (I : (y)^*) = (I : I^*).$$

Thus $\tau_{par} \subseteq \tau$. But we know that $\tau \subseteq \tau_{par}$. Thus $\tau = \tau_{par}$. □

Theorem 3.13 *In a one-dimensional local domain R with with an infinite residue field $(x)^* = ((x) : \tau)$, where τ is the test ideal of R.*

Proof. By Theorem 3.11, we know $\tau = \mathfrak{c}$. We need only show that

$$(x) : \mathfrak{c} \subseteq (x)^* = \overline{(x)}.$$

Let $y \in (x) : \mathfrak{c}$. Then $\mathfrak{c}y \subseteq (x)$. We want to show that $y \in \overline{(x)}$. If we use the determinant trick, i.e., if we show that $\mathfrak{c}y \subseteq \mathfrak{c}x$ then we are done. Let $J = (\mathfrak{c}y : x)$. If we can show that $J \subseteq \mathfrak{c}$ i.e., if for $s \in \overline{R}$ then $sJ \subseteq J$, then

$$\mathfrak{c}y = \mathfrak{c}y \cap (x) = Jx$$

implies that $\mathfrak{c}y \subseteq \mathfrak{c}x$. But $z \in J$ implies $zx = cy$ where $c \in \mathfrak{c}$. But $szx = scy = xa$ for some $a \in R$ which gives us that $sz \in R$ which implies that $sJ \subseteq J$. □

Remark 3.14 *A complete local domain R with test ideal equal to R is isomorphic to $k[[t]]$.*

Proof. Note in a complete Gorenstein local domain R with test ideal equal to R, Theorem 3.13 implies that for every nonzero $x \in R$ then $(x) = (x)^*$. Hence by Theorem 1.11, R is F-regular and hence regular since $(x) = (x)^* = \overline{(x)}$ implies that R is normal. By the Cohen Structure Theorem $R \cong k[[t]]$. □

Using Theorem 3.13, we can also characterize all one-dimensional complete Gorenstein local domains with test ideal equal to the maximal ideal.

Theorem 3.15 *All complete one-dimensional local Gorenstein domains $(R, \mathfrak{m})$ with the test ideal equal to the maximal ideal, $\mathfrak{m}$, with algebraically closed residue field k of characteristic greater than or equal to 5 are isomorphic to $k[[t^2, t^3]]$.*

Proof. Let R be such a ring. Since R is Gorenstein we know that for any minimal reduction 1.3 (y) of the maximal ideal $\mathfrak{m}$,

$$(y) : \mathfrak{m} = (x, y)$$

where $x \notin (y)$. Thus

$$(y)^* = \overline{(y)} = \mathfrak{m} = (y, x)$$

and

$$\mathfrak{m}(y)^* = \mathfrak{m}^2 \subseteq (y).$$

Thus $l(R/(y)) \leq 2$. But R is not regular, thus $l(R/(y)) = 2$. Since R is Cohen-Macaulay and y is a maximal R-sequence, the multiplicity of the ring R, denoted by $\mathrm{e}(\mathfrak{m})$ is given by

$$\mathrm{e}(\mathfrak{m}) = \mathrm{e}((y)) = l(R/(y)) = 2.$$

Since R is complete, the Cohen Structure Theorem tells us that R is finite over $k[[y]]$. Since $\mathrm{e}(\mathfrak{m}) = 2$, $\{1, x\}$ form a basis of R over $k[[y]]$. Thus we can write $x^2 = b_1 x + b_2$ where b_1 and $b_2 \in k[[y]]$. Setting $a_1 = -b_1$ and $a_2 = -b_2$ then $x^2 + a_1 x + a_2 = 0$. Set

$$f = x^2 + a_1 x + a_2.$$

Replacing $x - a_1/2$ by x we see

$$f = x^2 - a$$

where $a \in k[[y]]$. Also by the Cohen Structure Theorem, we know R is a quotient of $k[[x, y]]$ and since $k[[x, y]]/(f)$ is a domain then $R \cong k[[x, y]]/(f)$. Note $a \in (y)$ otherwise f is a unit. Thus $f = x^2 - \alpha y^i$ where α is a unit.

We show if $i > 3$, then y is not in $\mathfrak{c}$. We will break this claim down into two parts. First note that R is a domain, so f is irreducible. First assume $i = 2n$ is even. Since k is algebraically closed and the characteristic is not 2, square roots exist in k. Thus

$$x = \pm\alpha^{1/2}y^r,$$

contradicting our assumption that f is irreducible. Now assume that $i = 2n+1$ is odd. Then $\alpha^{1/2}y^{r+1}$ is a root of the monic polynomial $z^2 - yx^2$ over $k[[x,y]]/(x^2 - \alpha y^{2r+1})[z]$. Thus $\alpha^{1/2}y^{r+1} \in \overline{(x)}$. Suppose $y \in \mathfrak{c}$. Then $y\alpha^{1/2}y^{r+1} \in (x)$. So

$$\alpha^{1/2}y^{r+2} - bx = c(x^2 - \alpha y^{2r+1}).$$

Rewriting this equality we get

$$\alpha^{1/2}y^{r+2} + c\alpha y^{2r+1} = bx + cx^2.$$

We know that $y^{2r+1} \in (x)$; thus the right side of this equality factors if $n+2 \geq 2n+1$ i.e. $3 \geq i$. Note that $i \neq 1$ otherwise R is regular. Thus $R = k[[x,y]]/(x^2 - \alpha y^3)$.

Using Hensel's lemma $z^3 - \alpha$ has distinct roots since the characteristic is not 3. Thus we can write $\alpha = \beta^3$ with β a unit and we see that

$$R = k[[x,y]]/(x^2 - (\beta y)^3) \cong k[[t^2,t^3]].$$

□

3.3 Two-dimensional Gorenstein normal domains

A main thrust in understanding test elements and test ideals in Gorenstein isolated singularities is to try to link the type of singularity with the test ideal. Recently Hara [11] used geometric methods to show that this tie exists. Before we state his main result we note the following definition:

Definition 3.16 *[12] Suppose $g : W' \to W$ is a proper birational morphism of nonsingular varieties W' and W. Let $V \subseteq W$ be a subvariety. We say that $g^{-1}(V)$ is a* ***divisor with normal crossings*** *if $g^{-1}(V)$ is nonsingular and when r irreducible components $Y_1, \ldots, Y_r$ meet in a point P, then the equations $f_1, \ldots, f_r$ of the Y_i form part of a regular system of parameters at P.*

Theorem 3.17 *[11] Let R be a local Cohen Macaulay normal isolated singularity over a field of characteristic 0 and X a desingularization of Spec(R) with normal crossings. Then $H^{d-1}(X, \mathcal{O}_X) \cong 0^*_{H^d_{\mathfrak{m}}(R)}$.*

Remark 3.18 *In the case when R is a 2-dimensional complete Gorenstein isolated singularity $H^1(X, \mathcal{O}_X) \cong R/\tau$.*

Proof. Recall that the test ideal is equal to $\mathrm{Ann}_R(0^{*f.g.}_E)$ by 1.15 where E is the injective hull of $R/\mathfrak{m}$. But we know in a Gorenstein ring that $E = H^d_{\mathfrak{m}}(R)$. Thus

$$\tau = \mathrm{Ann}_R(0^{*\,f.g.}_{H^d_{\mathfrak{m}}(R)}).$$

Smith notes in [29, 3.3] that

$$0^{*}_{H^d_{\mathfrak{m}}(R)}{}^{f.g.} = 0^{*}_{H^d_{\mathfrak{m}}(R)}.$$

She also shows in [29, 3.1] that

$$\mathrm{Ann}_R(0^{*}_{H^d_{\mathfrak{m}}(R)}) = \mathrm{Ann}_R(0^{*}_{H^d_{\mathfrak{m}}(R)}{}^{\vee})$$

where $^{\vee}$ denotes the Matlis dual. Smith also shows in [29, 3.1] that

$$0^{*}_{H^d_{\mathfrak{m}}(R)} = (0^{*}_{H^d_{\mathfrak{m}}(R)}{}^{\vee})^{\vee} = R/\mathrm{Ann}_R(0^{*}_{H^d_{\mathfrak{m}}(R)}{}^{\vee}) = R/\tau.$$

□

We use the tight closure Briançon-Skoda theorem, which follows, to break down the classification of normal Gorenstein domains with test ideal equal to the maximal ideal into double points and triple points.

Theorem 3.19 ***Tight Closure Briançon-Skoda Theorem*** *[14] Let R be a Noetherian local ring of characteristic $p > 0$. Let I be an ideal in R generated by n elements. Then*

$$\overline{I^{r+r}} \subseteq (I^r)^*$$

for all $r \geq 1$.

Thus if $d = 2$ and y, z is a system of parameters which is a minimal reduction of the maximal ideal then the above Theorem says $\mathfrak{m}^2 \subseteq (y, z)^*$. Note $\tau\mathfrak{m}^2 \subseteq (y, z)$. If $\tau = \mathfrak{m}$ then $\mathfrak{m}^3 \subseteq (y, z)$. But $\mathfrak{m}^2 \subseteq (y, z)$ is possible so the classification breaks down into the two cases: 1) $\mathfrak{m}^2 \subseteq (y, z)$ and 2) $\mathfrak{m}^3 \subseteq (y, z)$ where $\mathfrak{m}^2 \not\subseteq (y, z)$. We will classify case 1) using the methods below but case 2) remains unsolved.

To simplify the proof of the classification when $\mathfrak{m}^2 \subseteq (y, z)$, we use the following Theorem to reduce the problem to hypersurfaces of the form $k[[x, y, z]]/(x^2 - a)$ where $a \in k[[y, z]]$.

Theorem 3.20 *Let $(R, \mathfrak{m})$ be a two-dimensional complete normal Gorenstein domain of characteristic $p > 2$ with test ideal contained in the maximal ideal. Suppose that $\mathfrak{m}^2 \subseteq (y, z)$ where (y, z) is a minimal reduction of the maximal ideal. Then $R \cong k[[x, y, z]]/(x^2 - a)$ where $a \in k[[y, z]]$.*

Proof. Since R is Gorenstein and (y, z) is irreducible we know $((y, z) : \mathfrak{m}) = (x, y, z)$ where $x \notin (y, z)$. $\mathfrak{m}^2 \subseteq (y, z)$ implies $l(R/(y, z)) = 2$. Since R is Cohen Macaulay and y, z is a maximal R-sequence, $e(\mathfrak{m}) = e((y, z)) = l(R/(y, z)) = 2$. Since R is complete, applying the Cohen Structure Theorem shows that R is finite over $k[[y, z]]$ with basis generated by $e(\mathfrak{m}) = 2$ elements. Thus $\{1, x\}$ form a basis of R over $k[[y, z]]$. We write $x^2 = b_1 x + b_2$ where b_1 and $b_2 \in k[[y, z]]$. Setting $a_1 = -b_1$ and $a_2 = -b_2$ then $x^2 + a_1 x + a_2 = 0$. Set

$$f = x^2 + a_1 x + a_2.$$

Replacing $x - a_1/2$ by x we see

$$f = x^2 - a$$

where $a \in k[[y,z]]$. Note that $k[[x,y,z]]/(f)$ is a two-dimensional complete normal Gorenstein ring contained in R by [7, Lemma 11.1]. The Cohen Structure Theorem also tells us that R is a two-dimensional quotient of the ring $k[[x,y,z]]$. Thus R is isomorphic to $k[[x,y,z]]/(f)$. □

We have formulated the following criterion to classify which complete Gorenstein normal domains have test ideal equal to the maximal ideal.

Theorem 3.21 *Let $R = k[[x,y,z]]/(x^2-a)$ with $a \in k[[y,z]]$ and k an algebraically closed of characteristic p. Then the following are equivalent:*

A) $\mathfrak{m} = \tau$,

B) There exists a minimal reduction (v,w) of $\mathfrak{m}$ such that

1) $\mathfrak{m} = (v,w)^*$ *and*

2) $(v^2,w^2)^* \subseteq (v,w)^2$.

C) For every minimal reduction (v,w) of $\mathfrak{m}$,

1) $\mathfrak{m} = (v,w)^*$ *and*

2) $(v^2,w^2)^* \subseteq (v,w)^2$.

Proof. $(C \Rightarrow B)$ requires no proof.

$(B \Rightarrow A)$ Suppose v,w satisfy 1) and 2). Since R is an isolated singularity then for some power n, v^r and w^r are test elements. By Proposition 3.2, $\tau = ((v^r,w^r) : (v^r,w^r)^*)$. Note that $\tau + (v,w) = ((v,w) : (v,w)^*)$. By 1) $((v,w) : (v,w)^*) = ((v,w) : \mathfrak{m}) = \mathfrak{m}$ since $x \in (v,w)^*$. Thus $x + a_1v + a_2w \in \tau$ where $a_1, a_2 \in R$.

Assuming 2) we know that $(v^2,w^2)^* \subseteq (v,w)^2$. Again $\tau + (v^2,w^2) = ((v^2,w^2) : (v^2,w^2)^*)$. Thus $\tau + (v^2,w^2) \supseteq ((v^2,w^2) : (v,w)^2) = (v,w)$. This implies that

$(v,w) \subseteq \tau$. Since $(x + a_1 v + a_2 w, v, w) = (x, v, w) = \mathfrak{m}$ then $\tau = \mathfrak{m}$.

$(A \Rightarrow C)$ If $\mathfrak{m}$ is the test ideal then for any minimal reduction (v,w) of $\mathfrak{m}$ $\mathfrak{m} = ((v,w) : (v,w)^*)$ by Proposition 3.2. Since R is Gorenstein and $v, w \in \mathfrak{m}$ are test elements we can conclude by Theorem 3.1 that

$$\mathfrak{m} = ((v,w) : \mathfrak{m}) = (v,w)^*.$$

Also after applying Theorem 3.1

$$(v^2, w^2)^* = ((v^2, w^2) : \mathfrak{m}) = (u, v^2, w^2)$$

where u is a generator for the socle of (v^2, w^2). Thus vu and $wu \in (v^2, w^2)$ or $u \in (v, w^2)$ and $u \in (v^2, w)$. Hence

$$u \in (v, w^2) \cap (v^2, w) = (v,w)^2.$$

□

When $(x^2 - a)$ is a quasihomogeneous polynomial then we can apply the following nice result of Huneke [19] to $R = k[[x,y,z]]/(x^2 - a)$ determine which elements lie in the tight closure of any system of parameters.

Theorem 3.22 *[19] Let R be the ring $k[x_o, \ldots, x_d]/(f)$. Assume that R is an isolated singularity which is quasihomogeneous, where k is a field of characteristic p. Assume that the partial derivatives $\frac{\partial f}{\partial x_i}$ form a system of parameters in R for $1 \le i \le d$. Further assume that $p > (d-1)(deg(f)) - \Sigma_{1 \le i \le d} deg(x_i)$. Let $y_1, \ldots, y_d$*

be a homogeneous system of parameters of degrees $a_1, \ldots, a_d$. Set $A = a_1 + \ldots a_d$. Then

$$(y_1, \ldots, y_d)^* = (y_1, \ldots, y_d) + R_{\geq A}.$$

Using the above Theorem, we see that an element $u \in S = k[x, y, z]/(x^2 - a)$ is in the tight closure of a homogeneous system of parameters v, w if $\deg(u) \geq \deg(v) + \deg(w)$. Since S is torsion-free, module-finite and generically smooth over $k[y, z]$, a regular domain we can apply Proposition [14, 6.9] to get the same inequality of degrees over $\hat{S} = R$. Hence in the quasihomogeneous case we have the following Corollary of Theorem 3.21:

Corollary 3.23 *Let $R = k[[x, y, z]]/(x^2 - a)$ be an isolated singularity defined by the quasihomogeneous polynomial $x^2 - a$ in $k[x, y, z]$. Assume that k is a field of characteristic $p > deg(x^2 - a) - deg(y) - deg(z)$. Then the test ideal is equal to the maximal ideal if and only if the following three inequalities hold:*

1) $deg(x) \geq deg(y) + deg(z)$,

2) $deg(xy) < deg(y^2) + deg(z^2)$ and

3) $deg(xz) < deg(y^2) + deg(z^2)$.

Proof. By Theorem 3.21 to show that the test ideal is the maximal ideal is equivalent to showing the following two properties hold for the minimal reduction y, z of the maximal ideal:

1) $(y, z)^* = \mathfrak{m}$ and

2) $(y^2, z^2)^* \subseteq (y, z)^2$.

Since $x^2 - a$ is a quasihomogeneous polynomial then we can apply Theorem 3.22 to the ring $k[x, y, z]/(x^2 - a)$ to compute which elements are in $(y, z)^*$ and $(y^2, z^2)^*$ over the ring $S = k[x, y, z]/(x^2 - a)$.

Denoting $(x, y, z) = \mathfrak{n} \subseteq S$ and $\hat{S}$ the $\mathfrak{n}$-adic completion of S, we note that $\hat{S} = R$. Thus by Proposition [14, 6.9], $(y, z)^*R = ((y, z)R)^*$ and $(y^2, z^2)^*R = ((y^2, z^2)R)^*$. Thus showing $x \in (y, z)^*$ and $xy, xz \notin (y^2, z^2)^*$ over the ring R is equivalent to showing $x \in (y, z)^*$ and $xy, xz \notin (y^2, z^2)^*$ over the ring S. By Theorem 3.22 this is equivalent to showing:

1) $\deg(x) \geq \deg(y) + \deg(z)$,

2) $\deg(xy) < \deg(y^2) + \deg(z^2)$ and

3) $\deg(xz) < \deg(y^2) + \deg(z^2)$. □

To apply Theorem 3.21 when $(x^2 - a)$ is not quasihomogeneous in practice we want to reduce our calculations to elements that lie in $k[[y, z]]$. Since $x^{q+1} = a^{\frac{q+1}{2}}$ and $a^{\frac{q+1}{2}} \in k[[y, z]]$ we can formulate Theorem 3.21 over $k[[y, z]]$ to read:

Remark 3.24 *Let $R = k[[x, y, z]]/(x^2 - a)$ with $a \in k[[y, z]]$ and k algebraically closed. Then the test ideal is the maximal ideal if and only if the following two properties hold over $k[[y, z]]$:*

1) $a^{\frac{q+1}{2}} \in (y^q, z^q)$.

2) $y^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ and $z^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$.

Proof. Combining Theorem 3.21 B and C we need only show that $x \in (y,z)^*$ and $(y^2,z^2)^* \subseteq (y,z)^2$. Recall in the proof of Theorem 3.21, that we may assume x is a test element. Thus we need only show that $x^{q+1} \in (y^q,z^q)$ and $x^{q+1}(\alpha^q y^q + \beta^q z^q) \notin (y^{2q},z^{2q})$ for any α and β in R. In this case, these statements translate into:

I) $a^{\frac{q+1}{2}} \in (y^q,z^q)$ and

II) $a^{\frac{q+1}{2}}(\alpha^q y^q + \beta^q z^q) \notin (y^{2q},z^{2q})$.

We note that if either α or β is zero this translates into 2). Now suppose neither α nor β is zero and presume $a^{\frac{q+1}{2}}(\alpha^q y^q + \beta^q z^q) \in (y^{2q},z^{2q})$. Then $a^{\frac{q+1}{2}} \in (y^{2q},z^{2q}) : (\alpha^q y^q + \beta^q z^q) = ((y^2,z^2) : (\alpha y + \beta z))^{[q]} = ((y,z)^2)^{[q]}$ which is a contradiction. □

Theorem 3.25 *Let $R = k[[y,z]]$ where k is algebraically closed of characteristic p. Let $a \in \mathfrak{m}$ be square-free. Then $a^{\frac{q+1}{2}} \in (y^q,z^q)$ for all large q if and only if after a change of variables $a \in \mathfrak{m}^4$ or a is of the form $y^3 + b$ where $b \in (yz^4) + \mathfrak{m}^6$.*

In proving Theorem 3.25 we need a few lemmas about binomial coefficients in characteristic p. We also refer to some Lemmas in the Appendix to determine analytic isomorphism classes of forms of low degree having terms of higher order.

Definition 3.26 *[9] Let $n(m)$ be defined to be the number of times that p divides $m!$, i.e., $n(m) = \lfloor \frac{m}{p} \rfloor + \lfloor \frac{m}{p^2} \rfloor + \ldots + \lfloor \frac{m}{p^k} \rfloor$ where p^{k+1} doesn't divide m.*

Lemma 3.27 *If $q = p^e$, $e > 1$ and $p \geq 5$ we can find an integer k such that $k < \frac{q-1}{4}$ and $\binom{\frac{q+1}{2}}{k} \not\equiv 0 \bmod p$ and $3(\frac{q+1}{2} - k) < q$.*

Proof. We break the proof down into two cases. First let $p \equiv 1 \bmod 4$. The first $k < \frac{q-1}{4}$ is $k = \frac{q-5}{4}$. We have $\binom{\frac{q+1}{2}}{\frac{q-5}{4}} = \frac{(\frac{q+1}{2})!}{(\frac{q-5}{4})!(\frac{q+7}{4})!}$. Suppose $q = p^e = 4k_e + 1$ and for general n, we have $p^r = 4k_r + 1$. $\binom{\frac{q+1}{2}}{\frac{q-5}{4}} \equiv 0 \bmod p$ if p divides $\binom{\frac{q+1}{2}}{\frac{q-5}{4}}$. This happens if $n(\frac{q+1}{2}) > n(\frac{q-5}{4}) + n(\frac{q+7}{4})$. Note

$$n(\frac{q+1}{2}) = \lfloor\frac{p^e+1}{2p}\rfloor + \lfloor\frac{p^e+1}{2p^2}\rfloor + \ldots + \lfloor\frac{p^e+1}{2p^{e-1}}\rfloor$$

$$= \lfloor\frac{p^{e-1}}{2}\rfloor + \lfloor\frac{p^{e-2}}{2}\rfloor + \ldots + \lfloor\frac{p}{2}\rfloor = 2k_{e-1} + \ldots + 2k_1$$

since $\frac{1+p^r}{2p^r} < 1$ for $p^r > 1$.

$$n(\frac{q-5}{4}) = \lfloor\frac{p^e-5}{4p}\rfloor + \lfloor\frac{p^e-5}{4p^2}\rfloor + \ldots + \lfloor\frac{p^e-5}{4p^{e-1}}\rfloor$$

$$= \lfloor\frac{p^{e-1}}{4}\rfloor + \lfloor\frac{p^{e-2}}{4}\rfloor + \ldots + \lfloor\frac{p}{4}\rfloor = k_{e-1} + \ldots + k_1$$

since $\frac{p^r-5}{4p^r} \geq 0$ for $p^r \geq 5$.

$$n(\frac{q+7}{4}) = \lfloor\frac{p^e+7}{4p}\rfloor + \lfloor\frac{p^e+7}{4p^2}\rfloor + \ldots + \lfloor\frac{p^e+7}{4p^{e-1}}\rfloor$$

$$= \lfloor\frac{p^{e-1}}{4}\rfloor + \lfloor\frac{p^{e-2}}{4}\rfloor + \ldots + \lfloor\frac{p}{4}\rfloor = k_{e-1} + \ldots + k_1$$

since $\frac{p^r+7}{4p^r} < 1$ for $7 < 3p^r$. Thus $n(\frac{q+1}{2}) = n(\frac{q-5}{4}) + n(\frac{q+7}{4})$ so p doesn't divide $\binom{\frac{q+1}{2}}{\frac{q-5}{4}}$. Also note for $q > 21$, we have $\frac{3(q+7)}{4} < q$.

Now suppose that $p \equiv 3 \bmod 4$. I will do the case $q \equiv 3 \bmod 4$, and leave $q \equiv 1 \bmod 4$ to the reader since it is basically the same. Note if n is odd then $p^r = 4k_r + 3$ and if n is even $p^r = 4k_r + 1$. The first k we can choose such that $k < \frac{q-1}{4}$ is $k = \frac{q-3}{4}$. But $\binom{\frac{q+1}{2}}{\frac{q-3}{4}} \equiv 0 \bmod p$ since

$$n(\frac{q+1}{2}) = \lfloor\frac{p^e+1}{2p}\rfloor + \lfloor\frac{p^e+1}{2p^2}\rfloor + \ldots + \lfloor\frac{p^e+1}{2p^{e-1}}\rfloor$$

$$= \lfloor \frac{p^{e-1}}{2} \rfloor + \lfloor \frac{p^{e-2}}{2} \rfloor + \ldots + \lfloor \frac{p}{2} \rfloor = \Sigma_{i=1}^{e-1} 2k_i + a_i$$

where

$$a_i = \begin{cases} 0 & \text{for } i \text{ even} \\ 1 & \text{for } i \text{ odd.} \end{cases}$$

$$n(\frac{q-3}{4}) = \lfloor \frac{p^e-3}{4p} \rfloor + \lfloor \frac{p^e-3}{4p^2} \rfloor + \ldots + \lfloor \frac{p^e-3}{4p^{e-1}} \rfloor$$
$$= \lfloor \frac{p^{e-1}}{4} \rfloor + \lfloor \frac{p^{e-2}}{4} \rfloor + \ldots + \lfloor \frac{p}{4} \rfloor = k_{e-1} + \ldots + k_1$$

since $\frac{p^r-3}{4p^r} \geq 0$ for $p^r \geq 1$.

$$n(\frac{q+5}{4}) = \lfloor \frac{p^e+5}{4p} \rfloor + \lfloor \frac{p^e+5}{4p^2} \rfloor + \ldots + \lfloor \frac{p^e+5}{4p^{e-1}} \rfloor$$
$$= \lfloor \frac{p^{e-1}}{4} \rfloor + \lfloor \frac{p^{e-2}}{4} \rfloor + \ldots + \lfloor \frac{p}{4} \rfloor = k_{e-1} + \ldots + k_1$$

since $\frac{p^r+5}{4p^r} > 1$ for $3p^r > 5$. Thus $n(\frac{q-3}{4}) + n(\frac{q+5}{4}) = 2k_{e-1} + \ldots + 2k_1 < n(\frac{q+1}{2})$. If we take $k = \frac{p^e - p^{e-1} + 2}{4}$, then $3(\frac{p^e+1}{2} - \frac{p^e - p^{e-1}+2}{4}) = 3(\frac{p^e + p^{e-1}}{4}) < p^e$. Also

$$n(\frac{p^e - p^{e-1} + 2}{4}) = \lfloor \frac{p^e - p^{e-1} + 2}{4p} \rfloor + \lfloor \frac{p^e - p^{e-1} + 2}{4p^2} \rfloor + \ldots + \lfloor \frac{p^e - p^{e-1} + 2}{4p^{e-1}} \rfloor$$
$$= \lfloor \frac{p^{e-1} - p^{e-2}}{4} \rfloor + \lfloor \frac{p^{e-2} - p^{e-3}}{4} \rfloor + \ldots + \lfloor \frac{p-1}{4} \rfloor = \Sigma_{i=1}^{e-1} k_i - k_{i-1} + a_i$$

where

$$a_i = \begin{cases} -1 & \text{for } i \text{ even} \\ 0 & \text{for } i \text{ odd.} \end{cases}$$

$$n(\frac{p^e + p^{e-1}}{4}) = \lfloor \frac{p^e + p^{e-1}}{4p} \rfloor + \lfloor \frac{p^e + p^{e-1}}{4p^2} \rfloor + \ldots + \lfloor \frac{p^e + p^{e-1}}{4p^{e-1}} \rfloor$$

$$= \lfloor\frac{p^{e-1}+p^{e-2}}{4}\rfloor + \lfloor\frac{p^{e-2}+p^{e-3}}{4}\rfloor + \ldots + \lfloor\frac{p-1}{4}\rfloor == \Sigma_{i=1}^{e-1} k_i + k_{i-1} + 1.$$

Thus

$$n(\frac{p^e-p^{e-1}-2}{4}) + n(\frac{p^e+p^{e-1}}{4}) = 2k_{e-1} + 2k_{e-2} + 1 + \ldots + 2k_1 + 1 = n(\frac{q+1}{2}).$$

□

Lemma 3.28 *For $p > 5$ and $e > 1$ there exists a k such that $\frac{3q+5}{10} < k < \frac{q}{3}$ where k is maximal with respect to the property $\binom{\frac{q+1}{2}}{k} \not\equiv 0 \bmod p$.*

Proof. Let $p \equiv 1 \bmod 6$. Then for all powers of n, $p^r \equiv 1 \bmod 6$. Denote $p^r = 6k_r + 1$. The first k we should check is $k = \frac{q-1}{3}$. Note $\binom{\frac{q+1}{2}}{\frac{q-1}{3}} = \frac{(\frac{q+1}{2})!}{(\frac{q-1}{3})!(\frac{q+5}{6})!} = 0$ if p divides $\binom{\frac{q+1}{2}}{\frac{q-1}{3}}$. This happens if $n(\frac{q+1}{2}) > n(\frac{q-1}{3}) + n(\frac{q+5}{6})$. Note

$$n(\frac{q+1}{2}) = \lfloor\frac{p^e+1}{2p}\rfloor + \lfloor\frac{p^e+1}{2p^2}\rfloor + \ldots + \lfloor\frac{p^e+1}{2p^{e-1}}\rfloor$$

$$= \lfloor\frac{p^{e-1}}{2}\rfloor + \lfloor\frac{p^{e-2}}{2}\rfloor + \ldots + \lfloor\frac{p}{2}\rfloor = 3k_{e-1} + \ldots + 3k_1,$$

$$n(\frac{q-1}{3}) = \lfloor\frac{p^e-1}{3p}\rfloor + \lfloor\frac{p^e-1}{3p^2}\rfloor + \ldots + \lfloor\frac{p^e-1}{3p^{e-1}}\rfloor$$

$$= \lfloor\frac{p^{e-1}}{3}\rfloor + \lfloor\frac{p^{e-2}}{3}\rfloor + \ldots + \lfloor\frac{p}{3}\rfloor = 2k_{e-1} + \ldots + 2k_1$$

and

$$n(\frac{q+5}{6}) = \lfloor\frac{p^e+5}{6p}\rfloor + \lfloor\frac{p^e+5}{6p^2}\rfloor + \ldots + \lfloor\frac{p^e+5}{6p^{e-1}}\rfloor$$

$$= \lfloor\frac{p^{e-1}}{6}\rfloor + \lfloor\frac{p^{e-2}}{6}\rfloor + \ldots + \lfloor\frac{p}{6}\rfloor = k_{e-1} + \ldots + k_1.$$

Thus $n(\frac{q+1}{2}) = n(\frac{q-1}{3}) + n(\frac{q+5}{6})$ so p doesn't divide $\binom{\frac{q+1}{2}}{\frac{q-1}{3}}$.

Now let $p \equiv 5 \bmod 6$. Since $p^{2r} \equiv 1 \bmod 6$ write $p^{2r} = 6k_{2r} + 1$. Similarly, $p^{2r+1} \equiv 5 \bmod 6$ so we denote $p^{2r+1} = 6k_{2r+1} + 5$. The first k we can choose such that $k < \frac{q}{3}$ is $k = \frac{q-1}{3}$ if e is even and $k = \frac{q-2}{3}$ if e is odd. But $\binom{\frac{q+1}{2}}{\frac{q-1}{3}} \equiv 0 \bmod p$ since

$$n(\frac{q+1}{2}) = \lfloor \frac{p^e+1}{2p} \rfloor + \lfloor \frac{p^e+1}{2p^2} \rfloor + \ldots + \lfloor \frac{p^e+1}{2p^{e-1}} \rfloor$$

$$= \lfloor \frac{p^{e-1}}{2} \rfloor + \lfloor \frac{p^{e-2}}{2} \rfloor + \ldots + \lfloor \frac{p}{2} \rfloor = \Sigma_{i=1}^{e-1} 3k_i + 2\lfloor \frac{e+1}{2} \rfloor,$$

$$n(\frac{q-1}{3}) = \lfloor \frac{p^e-1}{3p} \rfloor + \lfloor \frac{p^e-1}{3p^2} \rfloor + \ldots + \lfloor \frac{p^e-1}{3p^{e-1}} \rfloor$$

$$= \lfloor \frac{p^{e-1}}{3} \rfloor + \lfloor \frac{p^{e-2}}{3} \rfloor + \ldots + \lfloor \frac{p}{3} \rfloor = 2k_{e-1} + \ldots + 2k_1$$

and

$$n(\frac{q+5}{6}) = \lfloor \frac{p^e+5}{6p} \rfloor + \lfloor \frac{p^e+5}{6p^2} \rfloor + \ldots + \lfloor \frac{p^e+5}{6p^{e-1}} \rfloor$$

$$= \lfloor \frac{p^{e-1}}{6} \rfloor + \lfloor \frac{p^{e-2}}{6} \rfloor + \ldots + \lfloor \frac{p}{6} \rfloor = k_{e-1} + \ldots + k_1.$$

Thus $n(\frac{q-1}{3}) + n(\frac{q+5}{6}) = 3k_{e-1} + \ldots + 3k_1 < n(\frac{q+1}{2})$. If we can find any k between $\frac{3q+5}{10}$ and $\frac{q}{3}$ then there will exist a maximal k. Notice if $k = \dfrac{2p^e - p^{e-1} + 3}{6}$ then for $p > 5$, $\dfrac{2p^e - p^{e-1} + 3}{6} > \dfrac{3q+5}{10}$.

$$n(\frac{2p^e - p^{e-1} + 3}{6}) = \Sigma_{i=1}^{e-1}(2k_i - k_{i-1}) + (-1)^{i+1}.$$

And

$$n(\frac{p^e+1}{2} - \frac{2p^e - p^{e-1} + 3}{6}) = n(\frac{p^e + p^{e-1}}{6}) = \Sigma_{i=1}^{e-1} k_i + k_{i-1} + 1.$$

Thus

$$n(\frac{2p^e - p^{e-1} + 3}{6}) + n(\frac{p^e + p^{e-1}}{6}) =$$

$$\Sigma_{i=1}^{e-1}(3k_i) + e + (-1)^{e+1} = \Sigma_{i=1}^{e-1}(3k_i) + 2\lfloor\frac{e+1}{2}\rfloor.$$

□

One more lemma of the same nature which we will exploit later is the following:

Lemma 3.29 *Suppose $p > 3$ then $\binom{q-1}{\frac{q+1}{2}} \not\equiv 0$ mod p.*

Proof. Since $q = p^e \equiv 1$ mod 2 there exists an integer k_e such that $p^e = 2k_e + 1$. $\binom{q-1}{\frac{q+1}{2}} = \frac{(q-1)!}{(\frac{q+1}{2})!(\frac{q-3}{2})!}$. Note $\binom{q-1}{\frac{q+1}{2}} \equiv 0$ mod p if p divides $\binom{q-1}{\frac{q+1}{2}}$. This happens if $n(q-1) > n(\frac{q+1}{2}) + n(\frac{q-3}{2})$. Note

$$n(q-1) = \lfloor\frac{p^e-1}{p}\rfloor + \lfloor\frac{p^e-1}{p^2}\rfloor + \ldots + \lfloor\frac{p^e-1}{p^{e-1}}\rfloor$$

$$= \lfloor p^{e-1} - \frac{1}{p}\rfloor + \lfloor p^{e-2} - \frac{1}{p^2}\rfloor + \ldots + \lfloor p - \frac{1}{p^{e-1}}\rfloor$$

$$= \lfloor 2k_{e-1} + 1 - \frac{1}{p}\rfloor + \lfloor 2k_{e-2} + 1 - \frac{1}{p^2}\rfloor + \ldots + \lfloor 2k_1 + 1 - \frac{1}{p^{e-1}}\rfloor = 2k_{e-1} + \ldots + 2k_1$$

since $\frac{1}{p^n} < 1$ for $p^r > 1$.

$$n(\frac{q+1}{2}) = \lfloor\frac{p^e+1}{2p}\rfloor + \lfloor\frac{p^e+1}{2p^2}\rfloor + \ldots + \lfloor\frac{p^e+1}{2p^{e-1}}\rfloor$$

$$= \lfloor\frac{p^{e-1}}{2}\rfloor + \lfloor\frac{p^{e-2}}{2}\rfloor + \ldots + \lfloor\frac{p}{2}\rfloor = k_{e-1} + \ldots + k_1$$

since $1 > \frac{p^n+1}{2p^n} \geq 0$ for $p > 1$ for all $n \in \mathbb{N}$.

$$n(\frac{q-3}{2}) = \lfloor\frac{p^e-3}{2p}\rfloor + \lfloor\frac{p^e-3}{2p^2}\rfloor + \ldots + \lfloor\frac{p^e-3}{2p^{e-1}}\rfloor$$

$$= \lfloor\frac{p^{e-1}}{2}\rfloor + \lfloor\frac{p^{e-2}}{2}\rfloor + \ldots + \lfloor\frac{p}{2}\rfloor = k_{e-1} + \ldots + k_1$$

since $\dfrac{p^r - 3}{2p^r} < 1$ for $p > 3$. Thus $n(q-1) = n(\frac{q+1}{2}) + n(\frac{q-3}{2})$ so p doesn't divide $\binom{q-1}{\frac{q+1}{2}}$. □

Proof. (Theorem 3.25) Note for any $a \in (y,z)^4, a^{\frac{q+1}{2}} \subseteq (y^q, z^q)$, since

$$((y,z)^4)^{\frac{q+1}{2}} = (y,z)^{2(q+1)} \subseteq (y^q, z^q).$$

Suppose $a \notin (y,z)^4$. Then by elimination of all other possibilities we need to show that $a = y^3 + b$ where $b \in (yz^4) + (y,z)^6$.

First consider the case where a is of the form a quadratic plus an element $b \in (y,z)^3$. After a change of variables we can rewrite the quadratic in the form yz or y^2.

By Lemma A.1 we can rewrite $yz + b = y'z'$. Thus after a change of variables we can treat all elements of the form $yz + b$ as yz. But $(yz)^{\frac{q+1}{2}} \notin (y^q, z^q)$. Hence this case is eliminated.

By Lemma A.2 we can rewrite $y^2 + b$ as $y^2 + cz^3$ where c is either a unit or a unit times z^i where $i > 1$. Thus after a change of variables we may assume $y^2 + b = y^2 + cz^3$. The first term in the expansion of $(y^2 + cz^3)^{\frac{q+1}{2}}$ with the exponent of y less than q is $(\frac{q+1}{2})y^{q-1}cz^3$. But $y^{q-1}z^3 \notin (y^q, z^q)$. If c is not a unit, then Lemma A.2 guarantees that $y^2 + cz^3 = y^2 + dz^i$ where d is a unit and $i \geq 4$. Note that i must be finite; otherwise, y^2 is not square-free. As in the case $y^2 + cz^3$ we can choose $q > i$ and we again see that $y^{q-1}z^i \notin (y^q, z^q)$. Now this case is also eliminated.

Now consider the case where a is a cubic plus an element $b \in (y,z)^4$. After a change of variables we can write the cubic in one of the three following forms: $yz(y+z), y^2z, y^3$.

By Lemma A.3 we can rewrite $yz(y+z)+b$ as $yz(y+z)$. The first term where the exponent of y is less than q in the expansion of $(yz(y+z))^{\frac{q+1}{2}}$ is $\binom{(\frac{q+1}{2})}{2} y^{q-1} z^{\frac{q+3}{2}}$. By Lemma 3.27 $\binom{(\frac{q+1}{2})}{2} \not\equiv 0 \bmod p$ and $y^{q-1}z^{\frac{q+3}{2}} \notin (y^q, z^q)$ for $q > 3$. This case is now eliminated.

By Lemma A.4 we can rewrite $y^2z + b$ as $y^2z + cz^4$ where c is either a unit or a unit times z^i for $i > 1$. For c a unit, the first term in the expansion of $(y^2z + cz^4)^{\frac{q+1}{2}}$ where the exponent of y is less than q is $(\frac{q+1}{2}) y^{q-1} z^{\frac{q+7}{2}}$. And for $q > 7, y^{q-1}z^{\frac{q+7}{2}} \notin (y^q, z^q)$. If z divides c, then by Lemma A.4 we can find a finite i such that $y^2z + cz^4 = y^2z + dz^i$ and d is a unit. Now for $q > 2i - 1$, $y^{q-1}z^{\frac{q+2i-1}{2}} \notin (y^q, z^q)$. This case is also eliminated.

Suppose that $a = y^3 + b$ where $b \in (y,z)^4$. By Lemma A.5 after a change of variables we can rewrite $y^3 + b$ as $y^3 + uz^3$ where $u \in (y,z)$. Assume that $u \in (y,z) \backslash (y,z)^2$. If $(y^3 + uz^3)^{\frac{q+1}{2}} \in (y^q, z^q)$, then the monomials in the expansion of

$$(y^3 + uz^3)^{\frac{q+1}{2}}$$

must be in (y^q, z^q). These monomials have the form

$$\binom{\frac{q+1}{2}}{k} y^{3k} u^{\frac{q+1}{2}-k} z^{3(\frac{q+1}{2}-k)}.$$

By Lemma 3.28 we can find a largest k, such that $\binom{\frac{q+1}{2}}{k} \not\equiv 0 \bmod p$ and $\frac{3q+5}{10} < k < \frac{q}{3}$. Assume

$$y^{3k}u^{\frac{q+1}{2}-k}z^{3(\frac{q+1}{2}-k)} \in (y^q, z^q) + (y,z)^{2q+3-k}.$$

Then

$$u^{\frac{q+1-2k}{2}} \in (y^{q-3k}, z^{\frac{6k-q-3}{2}}) + (y,z)^{\frac{q+3-2k}{2}}.$$

But

$$\frac{q+1-2k}{2} < \frac{6k-q-3}{2} \text{ for } \frac{q+2}{4} < \frac{3q+5}{10} < k.$$

Also note that

$$q - 3k < \frac{q+1-2k}{2} \text{ for } \frac{q-1}{4} < \frac{3q+5}{10} < k.$$

So if $u \in (y,z)\backslash(y,z)^2$ then

$$u \in (y, z^{\lfloor\frac{6k-q-3}{q+1-2k}\rfloor+1}) + (y,z)^2.$$

In otherwords, $u \in (y, z^2)$. Now rewrite $y^3 + uz^3$ as $y(y^2+z^3) + dz^5$. First let us look at the expansion of

$$(y(y^2+z^3))^{\frac{q+1}{2}}.$$

By Lemma 3.27 we can find $k < \frac{q-1}{4}$ such that $\binom{\frac{q+1}{2}}{k} \not\equiv 0 \bmod p$ and $\frac{3q+3-6k}{2} < q$. Thus

$$y^{\frac{q+1+4k}{2}}z^{\frac{3q+3-6k}{2}} \notin (y^q, z^q).$$

We need to make sure that the z^5 term does not change the coefficient of $y^{\frac{q+1+4k}{2}}z^{\frac{3q+3-6k}{2}}$. We need to look at the expansion of

$$z^{5i}(y(y^2+z^3))^{\frac{q+1-2i}{2}}$$

for any $i < \frac{q+1}{2}$. We want to find an l such that $5i+3l = \frac{3q+3-6k}{2}$. So $l = \frac{3q+3-10i-6k}{6}$. With such an l, the exponent of z will be the same. We need to see for which i the exponent of y will be the same. The exponent of y will be $\frac{3q+3+2i+12k}{6}$. Setting $\frac{3q+3+2i+12k}{6} = \frac{q+1+4k}{2}$ we see that $i = 0$. Thus

$$(y(y^2+z^3)+dz^5)^{\frac{q+1}{2}} \notin (y^q, z^q).$$

Thus we have eliminated the case when $a = y^3 + b$ where $b \in (y,z)^4$.

Now suppose that $a = y^3 + b$ where $b \in (y,z)^5$. Again use Lemma A.5 to rewrite $y^3 + b$ as $y^3 + uz^4$ after a change of variables where $u \in (y,z)\backslash(y,z)^2$. If $(y^3+uz^4)^{\frac{q+1}{2}} \in (y^q, z^q)$, then the monomials in the expansion of

$$(y^3+uz^4)^{\frac{q+1}{2}}$$

must be in (y^q, z^q). These monomials have the form

$$\binom{\frac{q+1}{2}}{k} y^{3k} u^{(\frac{q+1}{2}-k)} z^{4(\frac{q+1}{2}-k)}.$$

By Lemma 3.28 we can find a largest k, such that $\binom{\frac{q+1}{2}}{k} \not\equiv 0 \bmod p$ is nonzero and $\frac{3q+5}{10} < k < \frac{q}{3}$. Assume

$$y^{3k} u^{\frac{q+1}{2}-k} z^{4(\frac{q+1}{2}-k)} \in (y^q, z^q) + (y,z)^{q+2-k}.$$

Then

$$u^{\frac{q+1-2k}{2}} \in (y^{q-3k}, z^{\frac{8k-2q-4}{2}}) + (y,z)^{\frac{q+3-2k}{2}}.$$

But

$$\frac{q+1-2k}{2} < \frac{8k-2q-4}{2} \text{ for } \frac{3q+5}{10} < k.$$

Again note that

$$q - 3k < \frac{q+1-2k}{2} \text{ for } \frac{q-1}{4} < \frac{3q+5}{10} < k.$$

So if $u \in (y,z)\backslash(y,z)^2$ then

$$u \in (y, z^{\lfloor \frac{8k-2q-4}{q+1-2k} \rfloor + 1}) + (y,z)^2.$$

In otherwords, $u \in (y, z^2)$. Now rewrite $y^3 + uz^4$ as $y(y^2 + z^4) + dz^6$. Now note in the expansion of

$$(y(y^2 + z^4) + dz^6)^{\frac{q+1}{2}}$$

that monomials will be in the form

$$y^{i+2l} z^{3q+3-6i+4i-4l} = y^{i+2l} z^{3q+3-2i-4l}.$$

For the exponent of y, $i + 2l$, to be less than q, the exponent of z will be greater than $q + 3$. Thus $(y(y^2 + z^4) + dz^6)^{\frac{q+1}{2}} \in (y^q, z^q)$.

Now we only need to check that $y^3 + b$ where $b \in (y,z)^6$ satisfy the Theorem. Again use Lemma A.5 to rewrite $y^3 + b$ as $y^3 + uz^5$ where $u = (cy + dz) \in (y,z)$. When we expand

$$(y(y^2 + cz^5) + dz^6)^{\frac{q+1}{2}}$$

we want all monomials to be in (y^q, z^q). The monomials are in the form

$$y^{i+2l} z^{3q+3-6i+5i-5l} = y^{i+2l} z^{3q+3-i-5l}.$$

If the exponents of y, $i + 2l$, are less than q then the exponent of z will be greater than $q + 3$. Thus if $a = y^3 + b$ where $b \in (y,z)^6$ then $a^{\frac{q+1}{2}} \in (y^q, z^q)$. $\square$

Let $R = k[[x,y,z]]/(x^2 - a)$ with $a \in (y,z) \bigcap k[[y,z]]$. We want to classify all such rings that have test ideal equal to the maximal ideal. Using Theorem 3.21, we need to look at $(y,z)^*$ and $(y,z^2)^*$ and $(y^2,z)^*$. We will break the proof down into several steps. By Theorem 3.25 we know that $x \in (y,z)^*$ if $a = y^3 + b$ where $b \in (yz^4)+(y,z)^6$ or $a \in (y,z)^4$ where (y,z) is seen as an ideal of $k[[y,z]]$. Thus we need to find polynomials $f = x^2 - a$ that also satisfy the second condition of Theorem 3.21 we need to look at $a \in (y,z)^3 \backslash (y,z)^4$, $a \in (y,z)^4 \backslash (y,z)^5$ and $a \in (y,z)^5 \backslash (y,z)^6$. Note that if $a \in (y,z)^6$ then $a^{\frac{6(q+1)}{2}} = a^{3(q+1)} \in (y^q, z^{2q})$ and (y^{2q}, z^q).

To find the isomorphism classes of a for which $k[[x,y,z]]/(x^2 - a)$ have the test ideal equal to the maximal ideal we need to invoke the Weierstrass Preparation Theorem [24]:

Theorem 3.30 *Let $(R, \mathfrak{m})$ be a complete local ring and let $f(X) \in R[[X]]$ be of the form $f(X) = \Sigma a_i X^i$ where $a_i \in \mathfrak{m}$ for $i \neq n$ and a_r is a unit. Then $f(X) = u(X^r + b_{r-1}X^{r-1} + \ldots + b_0)$ where $b_i \in \mathfrak{m}$ and u is a unit in $R[[X]]$.*

In a few instances we show that two forms are isomorphic by a Theorem of Cutkosky and Srinivasan [3] for which we recall the Jacobian of a form. Let $f \in k[[x_1, x_2, \ldots, x_d]]$. Define the **Jacobian of** f, $\Delta(f) = (f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d})$. Their Theorem [3, Theorem B] states that there exists an automorphism taking f to g in $k[[x_1, x_2, \ldots, x_d]]$ if $f \equiv g \mod \mathfrak{m}(\Delta(f))^2$.

3.4 Classification – the case when $a \in (y,z)^3 \backslash (y,z)^4$

In the classification of $R = k[[x,y,z]]/(x^2 - a)$ where $a = y^3 + b$ with $b \in (y,z)^{r+1} \backslash (y,z)^{r+2}$, we need to determine the isomorphism classes corresponding with each b. We determined by Lemma A.5 that we can replace $a = y^3 + b$ with $a = y^3 + cz^r$ where $c \in (y,z) \cap k[[y,z]]$ and $n \geq 3$. By the Weierstrass Preparation Theorem $y^3 + cz^i$ can be rewritten $u(y,z)(y^3 + f(z)yz^r + g(z)z^r)$ where $u(y,z)$ is a unit in $k[[y,z]]$ and $f(z)$ and $g(z)$ are elements of $k[[z]]$. Multiplying

$$(x^2 - u(y,z)(y^3 + f(z)yz^r + g(z)z^r))$$

by $u^{-1}(y,z)$ and as long as the characteristic of k is greater than 2 we can set $x_1 = u^{\frac{-1}{2}}(y,z)x$ and we see that

$$(x^2 - u(y,z)(y^3 + f(z)yz^r + g(z)z^r)) = u(y,z)(x_1^2 - (y^3 + f(z)yz^r + g(z)z^r)).$$

However since the ideals $(u(y,z)(x_1^2 - (y^3 + f(z)yz^r + g(z)z^r)))$ and $(x_1^2 - (y^3 + f(z)yz^r + g(z)z^r))$ are equal then

$$R = k[[x,y,z]]/(x^2 - u(y,z)(y^3 + f(z)yz^r + g(z)z^r))$$

is isomorphic to

$$R = k[[x,y,z]]/(x^2 - (y^3 + f(z)yz^r + g(z)z^r)).$$

To determine the distinct isomorphism classes determined by

$$(y^3 + f(z)yz^r + g(z)z^r)$$

we need to consider the three cases when:

i) $f(z) = 0$ and $g(z) \neq 0$,

ii) $f(z) \neq 0$ and $g(z) = 0$ and

iii) $f(z) \neq 0$ and $g(z) \neq 0$.

Note for nonzero $h(z) \in k[[z]]$ $h(z) = \alpha z^r$ for α a unit in $k[[z]]$. Thus i)-iii) break down into the following three cases:

i) $a = y^3 + \alpha z^i$, where $i > n$,

ii) $a = y^3 + \alpha y z^i$, where $i \geq n$ and

iii) $a = y^3 + \alpha y z^i + \beta z^j$, where $i \geq n$ and $j > n$.

The following lemma determines the only isomorphism classes for case i) and ii).

Lemma 3.31 *Suppose the characteristic of k is greater than i and let $R = k[[x,y,z]]/(x^2 - a)$ or $R = k[[x,y,z]]/(x^2 - a')$ where $a = y^3 + \alpha z^i$ and $a' = y^3 + \alpha y z^i$ and α is a unit in $k[[y,z]]$ in both cases. Then there exists an automorphism of $k[[x,y,z]]$ taking $x^2 - (y^3 + z^i)$ to $x^2 - a$ and $x^2 - (y^3 + yz^i)$ to $x^2 - a'$.*

Proof. Since the characteristic of R is greater than i and if α_k is the component of $\alpha \in k$ then $t^i - \alpha_k$ has a root in $k[t]$ Since $k[[x,y,z]]$ is complete then Hensel's lemma implies that $t^i - \alpha$ has a root in $k[[x,y,z]][t]$. Suppose that θ is a root of $t^i - \alpha$. Setting $z_1 = \theta z$, we can rewrite

$$y^3 + \alpha z^i = y^3 + z_1^i$$

and

$$y^3 + \alpha y z^i = y^3 + y z_1^i.$$

Thus the automorphism $s : k[[x,y,z]] \to k[[x,y,z]]$ given by $s(x) = x, s(y) = y$ and $s(z) = \theta z$ maps $(x^2 - (y^3 + z^i))$ to $(x^2 - a)$ and $(x^2 - (y^3 + y z^i))$ to $(x^2 - a')$. □

Case iii) is more complicated. We must consider the cases when $i \geq j$ and $i < j$. Note when $i \geq j$ we can rewrite $a = y^3 + \alpha y z^i + \beta z^j$ as $y^3 + (\beta + \alpha y z^{i-j}) z^j$ which is exactly case i) since $\beta + \alpha y z^{i-j}$ is a unit. When $i < j$ there will be several different isomorphism classes depending on how large j is in comparison with i. When $j = i + 1$ we have the following Lemma which follows similar reasoning to proofs found in Matsuki's classification of canonical, log canonical and terminal singularities [26]:

Lemma 3.32 *Suppose the characteristic of k is greater than $i + 1$ and let $R = k[[x,y,z]]/(x^2 - a)$ where $a = y^3 + \alpha y z^i + \beta z^{i+1}$ and α and β are units. Then there is an automorphism of $k[[x,y,z]]$ taking $x^2 - (y^3 + z^{i+1})$ to $x^2 - a$.*

Proof. Since the characteristic of R is greater than $i + 1$ and if β_k is the component of $\beta \in k$ then $t^{i+1} - \beta_k$ has a root in $k[t]$ Since $k[[x,y,z]]$ is complete then Hensel's lemma implies that $t^{i+1} - \beta$ has a root in $k[[x,y,z]][t]$. Suppose that θ is a root of $t^{i+1} - \beta$. Setting $z_1 = \theta z$ and $\alpha_1 = \frac{\alpha}{\theta^i}$, we can rewrite

$$y^3 + \alpha y z^i + \beta z^{i+1} = y^3 + \alpha_1 y z_1^i + z_1^{i+1}.$$

Let $s_1 : k[[x,y,z]] \to k[[x,y,z]]$ given by $s_1(x) = x, s_1(y) = y$ and $s_1(z) = \theta z$.

Now set $z_2 = z_1 + \frac{\alpha_1}{i+1}y$ and suppose α_2 and γ are the induced unit coefficients of $y^2 z_2^{i-1}$ and y^3 respectively. We can rewrite

$$y^3 + \alpha_1 y z_1^i + z_1^{i+1} = \gamma y^3 + \alpha_2 y^2 z_2^{i-1} + z_2^{i+1}.$$

Let $s_2 : k[[x,y,z]] \to k[[x,y,z]]$ given by $s_2(x) = x, s_2(y) = y$ and $s_2(z) = z + \frac{\alpha_1}{i+1}y$.

Set $y_1 = \gamma^{\frac{1}{3}} y$ and suppose α_3 is the induced unit coefficient of $y_1^2 z_2^{i-1}$. Since $i \geq 3$ then $2(i-1) \geq i+1$. Thus we can rewrite

$$\gamma y^3 + \alpha_2 y^2 z_2^{i-1} + z_2^{i+1} = y_1^3 + \alpha_3 y_1^2 z_2^{i-1} + z_2^{i+1}.$$

Let $s_3 : k[[x,y,z]] \to k[[x,y,z]]$ given by $s_3(x) = x, s_3(y) = \gamma^{\frac{1}{3}} y$ and $s_3(z) = z$.

Now set $y_2 = y_1 + \frac{1}{3}\alpha_3 z_2^{i-1}$ and suppose ϵ is the induced unit coefficient of z_2^{i+1}. We can rewrite

$$y_1^3 + \alpha_3 y_1^2 z_2^{i-1} + z_2^{i+1} = y_2^3 + \epsilon z_2^{i+1}.$$

Let $s_4 : k[[x,y,z]] \to k[[x,y,z]]$ given by $s_4(x) = x, s_4(y) = y + \frac{1}{3}\alpha_3 z^{i+1}$ and $s_4(z) = z$.

Setting $z_3 = (\epsilon)^{\frac{1}{i+1}} z_2$, we can rewrite

$$y_2^3 + \epsilon z_2^{i+1} = y_2^3 + z_3^{i+1}.$$

Let $s_5 : k[[x,y,z]] \to k[[x,y,z]]$ given by $s_5(x) = x, s_5(y) = y$ and $s_5(z) = (\epsilon)^{\frac{1}{i+1}} z$.

Thus the automorphism $s : k[[x,y,z]] \to k[[x,y,z]]$ given by

$$s = s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_5$$

maps $(x^2 - (y^3 + z^{i+1}))$ to $(x^2 - a)$. ☐

For $j \geq i$ we have the following Lemma:

Lemma 3.33 *Suppose the characteristic of k is greater than i and let $R = k[[x,y,z]]/(x^2 - a)$ where $a = y^3 + \alpha y z^i + \beta z^{i+j}$, α and β are units and $j \geq i$. Then there exists an automorphism of $k[[x,y,z]]$ taking $y^3 + yz^i$ to a.*

Proof. Setting $y_1 = y + \frac{\beta}{\alpha} z^j$, we can rewrite

$$y^3 + \alpha y z^i + \beta z^{i+j} = y_1^3 - \frac{3\beta}{\alpha} y_1^2 z^j + \frac{3\beta^2}{\alpha^2} y_1 z^{2j} - \frac{\beta^3}{\alpha^3} z^{3j} + \alpha y_1 z^i.$$

If we set $\gamma = (1 - \frac{3\beta}{\alpha} y_1^2 z^{j-i} + \frac{3\beta^2}{\alpha^2} y_1 z^{2j-i})$ we see that

$$y^3 + \alpha y z^i + \beta z^{i+j} = y_1^3 + \gamma y_1 z^i - \frac{\beta^3}{\alpha^3} z^{3j}.$$

Let $s_1 : k[[x,y,z]] \rightarrow k[[x,y,z]]$ be the automorphism defined by $s_1(x) = x$, $s_1(y) = y + \frac{\beta}{\alpha} z^j$ and $s_1(z) = z$.

Set $z_1 = \gamma^{\frac{1}{i}} z$ and suppose β_1 is the induced coefficient of z_1^{3j}. We can write

$$y_1^3 + \gamma y_1 z^i - \beta^3 z^{3j} = y_1^3 + y_1 z^i + \beta_1^3 z^{3j}.$$

Let $s_2 : k[[x,y,z]] \rightarrow k[[x,y,z]]$ be the automorphism defined by $s_2(x) = x$, $s_2(y) = y$ and $s_2(z) = \gamma^{\frac{1}{i}} z$.

But $3j > i + j$. Thus if we define $y_2 = y_1 + \beta_1 z^{3j-i}$ and proceed as above we can rewrite

$$y_1^3 + y_1 z^i + \beta_1^3 z^{3j} = y_2^3 + \gamma_1 y_2 z^i - \beta_1^3 z^{3(3j-i)}$$

where γ_1 is the induced coefficient of yz^i. Let $s_3 : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by $s_3(x) = x$, $s_3(y) = y + \beta_1 z^{3j-i}$ and $s_3(z) = z$.

Setting $z_2 = \gamma^{\frac{1}{i}} z_1$ and suppose β_2 is the induced coefficient of $z_1^{3(3j-i)}$. We can write

$$y_1^3 + \gamma_1 y z^i - \beta^3 z^{3(3j-i)} = y_1^3 + y_1 z^i + \beta_1^3 z^{3(3j-i)}.$$

Let $s_4 : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by $s_4(x) = x$, $s_4(y) = y$ and $s_4(z) = \gamma_1^{\frac{1}{i}} z$.

Again $3(3j-i) > 3j$ and $z^{3(3j-i)} \in \mathfrak{m}(\Delta(x^2-(y^3+yz^i)))^2$ Thus by [3, Theorem B], there exists an automorphism $s : k[[x,y,z]] \to k[[x,y,z]]$ taking $y^3 + yz^i$ to a. □

Theorem 3.34 *Suppose the characteristic of k is greater than 11 and let $R = k[[x,y,z]]/(x^2 - a)$ where $a \in (y,z)^3 \backslash (y,z)^4 \cap k[[y,z]]$, then $\mathfrak{m}$ is the test ideal if and only if after a change of variables we can write a in one of the following forms:*

1) $a = y^3 + z^i$, $6 \le i < 12$,

2) $a = y^3 + yz^i$, $4 \le i < 8$,

3) $a = y^3 + yz^i + \beta z^{i+j}$, *where β is a unit,* $4 \le i < 8$ *and* $2 \le j < i$,

$i = 8$ *and* $j = 2, 3$ *and* $i = 9$ *and* $j = 2$.

Proof. We need only show that $x \in (y,z)^*$ and $xy \notin (y^2,z^2)^*$ and $xz \notin (y^2,z^2)^*$ if and only if 1) - 3) hold by Theorem 3.21. By Theorem 3.25, $x \in (y,z)^*$ or equivalently $a^{\frac{q+1}{2}} \in (y^q, z^q)$ if and only if after a change of variables we can write

$a = y^3 + b$ where $b \in (yz^4) + (y,z)^6$ or $a \in (y,z)^4$. By Lemma A.5 after a change of variables we can replace $y^3 + b$ by $y^3 + cz^{i-1}$ if $b \in (y,z)^i \backslash (y,z)^{i+1}$ where $c \in (y,z)$. Note that by Remark 3.24 we need only show that $y^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ since $\frac{3(q+1)}{2} < 2q$ for $q > 3$ implies that $z^q y^{\frac{3(q+1)}{2}}$ which is a monomial in the expansion of $z^q a^{\frac{q+1}{2}}$ is not in (y^{2q}, z^{2q}) or in otherwords, $xz \notin (y^2, z^2)^*$.

As noted above after application of the Weierstrass Preparation Theorem we need only check which power series in the following three forms:

I) $a = y^3 + \alpha z^i$,

II) $y^3 + \alpha y z^i$ and

III) $y^3 + \alpha y z^i + \beta z^j$

have the property that $xy \notin (y^2, z^2)^*$.

I) As long as $\alpha \in k$ then $x^2 - (y^3 + \alpha z^i)$ is quasihomogeneous. Hence, we can apply Theorem 3.23 with the $\deg(x) = 3i, \deg(y) = 2i$, and $\deg(z) = 6$. Thus $xy \notin (y^2, z^2)^*$ if and only if $\deg(xy) = 5i < 4i + 12$. In otherwords, if and only if $i < 12$ and Theorem 3.25 implies that $i \geq 6$.

If $\alpha \notin k$ then $\alpha = \Sigma_{k,l \geq 0} \alpha_{kl} y^k z^l$. The monomials in the expansion of $(y^3 + \alpha z^i)^{\frac{q+1}{2}}$ are of the form

$$y^{\frac{5q+3}{2} + (k-3)r} z^{(i+l)r}.$$

Since $y^q (y^3 + \alpha_{00} z^i)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ for $6 \leq i < 12$ then there exists an integer r such that the coefficient of

$$y^{\frac{5q+3}{2} - 3r} z^{ir}$$

is not congruent to 0 modulo p and

$$y^{\frac{5q+3}{2}-3r}z^{ir} \notin (y^{2q}, z^{2q}).$$

As long as no other monomial in the expansion of $y^q(y^3+\alpha z^i)^{\frac{q+1}{2}}$ has the same exponents for both y and z then

$$y^q(y^3+\alpha z^i)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Suppose there exists $k, l > 0$ and some integer s such that

$$y^{\frac{5q+3}{2}-3r}z^{ir} = y^{\frac{5q+3}{2}+(k-3)s}z^{(i+l)s}.$$

This equality yields the following system of equations:

$$-3r = (k-3)s \text{ and } ir = (i+l)s.$$

Multiplying the first equation by i and the second by 3 and adding them together yields the following: $0 = (ik+3l)s$. But $k, l > 0$ implies $s = 0$. After substituting $s = 0$ back into the above equations, we have $r = 0$ which is a contradiction since

$$y^{\frac{5q+3}{2}} \in (y^{2q}, z^{2q}).$$

Thus $k = l = 0$ and $r = s$.

By Lemma 3.31, there exist an automorphism of $k[[x, y, z]]$ taking $y^3 + \alpha z^i$ to $y^3 + z^i$. Thus $y^3 + z^i$ where $6 \le i < 12$ determines an isomorphism class of rings with test ideal equal to the maximal ideal. Thus we have 1).

II) As long as $\alpha \in k$ then then

$$x^2 - (y^3 + \alpha y z^i)$$

is quasihomogeneous. We again apply Theorem 3.23, with the $\deg(x) = 3i, \deg(y) = 2i$, and $\deg(z) = 4$. Thus $xy \notin (y^2, z^2)^*$ if and only if $\deg(xy) = 5i < 4i + 8$. In otherwords, if and only if $i < 8$. But we were assuming from Theorem 3.25 that $i \geq 4$.

If $\alpha \notin k$ then $\alpha = \Sigma_{k,l \geq 0} \alpha_{kl} y^k z^l$. The monomials in the expansion of $(y^3 + \alpha y z^i)^{\frac{q+1}{2}}$ are of the form

$$y^{\frac{5q+3}{2} + (k-2)r} z^{(i+l)r}.$$

Since $y^q(y^3 + \alpha_{00} y z^i)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ for $6 \leq i < 12$ then there exists an integer r such that the coefficient of

$$y^{\frac{5q+3}{2} - 2r} z^{ir}$$

is not congruent to 0 modulo p and

$$y^{\frac{5q+3}{2} - 2r} z^{ir} \notin (y^{2q}, z^{2q}).$$

As long as no other monomial in the expansion of $y^q(y^3 + \alpha y z^i)^{\frac{q+1}{2}}$ has the same exponents for both y and z then

$$y^q(y^3 + \alpha z^i)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Suppose there exists $k, l > 0$ and some integer s such that

$$y^{\frac{5q+3}{2} - 2r} z^{ir} = y^{\frac{5q+3}{2} + (k-2)s} z^{(i+l)s}.$$

This equality yields the following system of equations:

$$-2r = (k-2)s \text{ and } ir = (i+l)s.$$

Multiplying the first equation by i and the second by 2 and adding them together yields the following: $0 = (ik + 2l)s$. But $k, l > 0$ implies $s = 0$. After substituting $s = 0$ back into the above equations, we have $r = 0$ which is a contradiction since

$$y^{\frac{5q+3}{2}} \in (y^{2q}, z^{2q}).$$

Thus $k = l = 0$ and $r = s$.

By Lemma 3.31, there exists an automorphism of $k[[x, y, z]]$ taking $y^3 + \alpha y z^i$ to $y^3 + yz^i$. Thus $y^3 + yz^i$ where $4 \leq i < 8$ determines an isomorphism class of rings with test ideal equal to the maximal ideal. Thus we have 2).

III) We need only consider the case when $i < j$ since Lemma 3.31 implies that if $i \geq j$ there exists an automorphism of $k[[x, y, z]]$ taking $x^2 - (y^3 + yz^i + \beta z^j)$ to $x^2 - (y^3 + z^j)$ which was case 1) for $6 \leq j < 12$. Also when $j = i + 1$, Lemma 3.32 implies there exists an automorphism of $k[[x, y, z]]$ taking $x^2 - (y^3 + yz^i + \beta z^j)$ $x^2 - (y^3 + z^j)$ which is also case 1) for $6 \leq j < 12$. Thus we need only consider $j > i + 1$. For simplicity replace j by $i + j$ where $j \geq 2$.

We show that the test ideal must be the maximal ideal by looking at the monomials in the expansion of

$$y^q(y^3 + \alpha y z^i + \beta z^{i+j})^{\frac{q+1}{2}}.$$

Taking into account that

$$\alpha = \Sigma_{m,r \geq 0} b_{mr} y^m z^r \text{ and } \beta = \Sigma_{l,k \geq 0} b_{lk} y^l z^k,$$

these monomials are in the form

$$y^{\frac{5q+3}{2}+(l-3)r+(m-2)s}z^{(i+j+k)r+(i+r)s},$$

for $0 \le r \le \frac{q+1}{2}$ and $0 \le s \le \frac{q+1}{2} - r$. Setting $r = 0$ gives us monomials in the expansion of

$$y^q(y^3 + \alpha y z^i)^{\frac{q+1}{2}}$$

and we know from II) that $xy \notin (y^2, z^2)^*$ for $4 \le i < 8$ and in particular there exists an integer s such that

$$y^{\frac{5q+3}{2}-2s}z^{is} \notin (y^{2q}, z^{2q})$$

and the coefficient of

$$y^{\frac{5q+3}{2}-2s}z^{is}$$

is not congruent to 0 modulo p. Thus as long as there is no other monomial with the same exponents for both y and z then

$$y^q(y^3 + \alpha y z^i + \beta z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Suppose not, then for some pair (t, u),

$$-2s = (l-3)t + (m-2)u \text{ and } is = (i+j+k)t + (i+n)u.$$

Multiplying the first equation by i and the second by 2 and then adding the two together yields

$$0 = (i(l-1) + 2(j+k))t + (im + 2n)u.$$

We assumed $k, l, m, n > 0$ thus $u = t = 0$ since $u, t \geq 0$. By plugging in $u = t = 0$ back into the original equations we see that $s = 0$ which is a contradiction. Thus either $m = n = 0$ or $(i(l-1) + 2(j+k)) = 0$. Since $j \geq 2$, $(i(l-1) + 2(j+k)) = 0$ if and only if $l = 0$ and $i = 2(j+k)$. Thus i must be even, i.e. $i = 4$ or $i = 6$.

When $i = 4$, this implies that $j + k = 2$. Since we are assuming that $j \geq 2$, this can only happen when $j = 2$ and $k = 0$. But $x^2 - (y^3 + yz^4 + z^6)$ is a quasihomogeneous polynomial with the following degrees: $\deg(x) = 12, \deg(y) = 8$, and $\deg(z) = 4$ satisfying $\deg(xy) = 20 < 16 + 8$. Thus by Theorem 3.23, $x \notin (y^2, z^2)^*$.

When $i = 6$, this implies that $j + k = 3$. Since we are assuming that $j \geq 2$, this can only happen when either $j = 2$ and $k = 1$ or $j = 3$ and $k = 0$. However in both of the above cases the monomials come from the expansion of $x^2 - (y^3 + yz^6 + z^9)$, and $x^2 - (y^3 + yz^6 + z^9)$ is a quasihomogeneous polynomial with the following degrees: $\deg(x) = 18, \deg(y) = 12$, and $\deg(z) = 4$ satisfying $\deg(xy) = 30 < 24 + 8$. Thus by Theorem 3.23, $x \notin (y^2, z^2)^*$.

When $j > \frac{i}{2}$, $(i(l-1) + 2(j+k)) > 0$ forces $t = 0$. Thus

$$y^q(y^3 + \alpha yz^i + \beta z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $4 \leq i < 8$ and $\frac{i}{2} \leq j < i$.

If $i = 7$ or 8 and $2 \leq j \leq 3$ or $i = 5, 6$ or 9 and $j = 2$ the monomials again will be in the form

$$y^{\frac{5q+3}{2}+(l-3)r+(m-2)s} z^{(i+j+k)r+(i+r)s},$$

for $0 \leq r \leq \frac{q+1}{2}$ and $0 \leq s \leq \frac{q+1}{2} - r$.

Suppose $\alpha, \beta \in k$ and $s = 0$, then the monomials come from the expansion of

$$y^q(y^3 + z^{i+j})^{\frac{q+1}{2}}$$

for which we know $xy \notin (y^2, z^2)^*$ for $i + j < 12$ which holds for all of the above cases. Moreover, there exists an integer s such that

$$y^{\frac{5q+3}{2}-3s} z^{(i+j)s} \notin (y^{2q}, z^{2q})$$

and the coefficient of

$$y^{\frac{5q+3}{2}-3s} z^{(i+j)s}$$

is not congruent to 0 modulo p. Thus as long as there is no other monomial with the same exponents for both y and z then

$$y^q(y^3 + \alpha y z^i + \beta z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Suppose not, then for some pair (t, u),

$$-3s = (l - 3)t + (m - 2)u \text{ and } (i + j)s = (i + j + k)t + (i + n)u.$$

Multiplying the first equation by $i + j$ and the second by 3 yields

$$0 = ((i + j)l + 3k)t + ((i + j)(m - 2) + 3(i + n))u$$

or $0 = ((i+j)l+3k)t+((m+1)i+(m-2)j+3n)u$. We know that $(i+j)l+3k \geq 0$ and for $i = 7$ or 8 and $2 \leq j \leq 3$ or $i = 5, 6$ or 9 and $j = 2$, $(m+1)i+(m-2)j+3n > 0$.

Thus since $t, u \geq 0$, then $u = 0$. Plugging $u = 0$ into the above equations forces $l = k = 0$ and $t = s$. Thus

$$y^q(y^3 + z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $i = 8, 2 \leq j \leq 3$ and $i = 9, j = 2$.

Note that if $j \geq i$, Lemma 3.33 implies that there exists an automorphism of $k[[x, y, z]]$ taking $x^2 - (y^3 + \alpha y z^i + \beta z^{i+j})$ to $x^2 - (y^3 + yz^i))$ which is case 2) for $4 \leq i < 8$. Thus $x^2 - (y^3 + \alpha y z^i + \beta z^{i+j})$ determines an isomorphism class for $4 \leq i < 8$ and $2 \leq j < i$, $i = 8, 2 \leq j \leq 3$ and $i = 9, j = 2$ which is case 3). □

3.5 Classification – the case when $a \in (y,z)^4 \backslash (y,z)^5$

When the a a quartic plus higher order terms there will be at least as many isomorphism classes as there are distinct linear factorizations of quartics. These factorizations are among the following:

A) a product of four independent linear terms, $yz(y+z)(y+\lambda z)$,

B) a product of a square and two independent linear terms, $y^2z(y+z)$,

C) a product of two independent squares, y^2z^2,

D) a product of a cube and an independent linear term, y^3z,

E) a fourth power of a linear term, y^4.

Using the Lemmas in the appendix if we have a quartic in one of the above forms plus higher order terms we can make a change of variables absorbing units into x to rewrite these quartics as:

A) $yz(y+z)(y+\lambda z)$, λ a unit,

B) $(y^2+cz^r)z(y+z)$, c a unit, $n \geq 3$,

C) $(y^2+cz^r)(z^2+dy^m)$, c and d units, $n, m \geq 3$,

D) $(y^3+cz^i)z$, $c \in (y,z)$ and $i \geq 3$,

E) $(y^2+cz^r)^2+dz^m$, c is either a unit or 0 and $d \in (y,z)$ and $m > n \geq 3$.

We can easily check that if R is defined by $k[[x,y,z]]/(x^2-a)$ where a is in the form A, B or C then $y^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ and $z^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$. In case D, we can use many of the lemmas in chapter 8 to analyze which $c \in (y,z)$ will force $y^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ and $z^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$. Case E is probably the most difficult to analyze because we need to consider the following six cases:

I) $y^4 + \alpha z^i$, where α is a unit in $k[[y,z]]$,

II) $y^4 + \alpha y z^i$, where α is a unit in $k[[y,z]]$,

III) $y^4 + \alpha y z^i + \beta z^j$, where α and β are units in $k[[y,z]]$,

IV) $(y^2 + z^i)^2 + \alpha z^r$, where α is a unit in $k[[y,z]]$,

V) $(y^2 + z^i)^2 + \alpha y z^r$, where α is a unit in $k[[y,z]]$,

VI) $(y^2 + z^i)^2 + \alpha y z^r + \beta z^m$, where α and β are units in $k[[y,z]]$.

The following lemma is useful in cases I-III:

Lemma 3.35 *Suppose the characteristic of k is greater than i and let $R = k[[x,y,z]]/(x^2 - a)$ or $R = k[[x,y,z]]/(x^2 - a')$ where $a = y^4 + \alpha z^i$ and $a' = y^4 + \alpha y z^i$ and α is a unit in $k[[y,z]]$ in both cases. Then there exists an automorphism of $k[[x,y,z]]$ taking $x^2 - (y^4 + z^i)$ to $x^2 - a$ and $x^2 - (y^4 + yz^i)$ to $x^2 - a'$.*

Proof. Since the characteristic of R is greater than i and if α_k is the component of $\alpha \in k$ then $t^i - \alpha_k$ has a root in $k[t]$. Since $k[[x,y,z]]$ is complete then Hensel's lemma implies that $t^i - \alpha$ has a root in $k[[x,y,z]][t]$. Suppose that θ is a root of $t^i - \alpha$. Setting $z_1 = \theta z$, we can rewrite

$$y^4 + \alpha z^i = y^4 + z_1^i$$

and

$$y^4 + \alpha y z^i = y^4 + y z_1^i.$$

Thus the automorphism $s : k[[x, y, z]] \to k[[x, y, z]]$ given by $s(x) = x, s(y) = y$ and $s(z) = \theta z$ maps $(x^2 - (y^4 + z^i))$ to $(x^2 - a)$ and $(x^2 - (y^4 + yz^i))$ to $(x^2 - a')$.

□

In case IV we observe that if $a = (y^2 + z^3)^2 + \alpha z^5$ then there exists an automorphism taking a to $y^4 + yz^4 + \beta z^5$ where β is defined in the following lemma.

Lemma 3.36 *Let k be an algebraically closed field of characteristic $p > 5$ and suppose $R = k[[x, y, z]]/(x^2 - a)$ where $a = (y^2 + z^3)^2 + \alpha z^5$, α a unit. Then there exists an automorphism of $k[[x, y, z]]$ taking $y^4 + yz^4 + \beta z^5$ to a where β is a unit defined in the proof.*

Proof. Set $z_1 = (\alpha + z)^{\frac{1}{5}} z$ and $\alpha_1 = \frac{2}{(\alpha+z)^{\frac{1}{5}}}$ then we can rewrite

$$(y^2 + z^3)^2 + \alpha z^5 = y^4 + \alpha_1 y^2 z_1^3 + z_1^5.$$

Let $s_1 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_1(x) = x$, $s_1(y) = y$ and $s_1(z) = (\alpha + z)^{\frac{1}{5}} z$.

Set $z_2 = z_1 + (\frac{-\alpha_1}{10})^{\frac{1}{2}} y$ and suppose that γ, δ and α_2 are the unit coefficients of $y^4, y^3 z_2^2$ and yz_2^4 then

$$y^4 + \alpha_1 y^2 z_1^3 + z_1^5 = \gamma y^4 + \delta y^3 z_2^2 + \alpha_2 y z_2^4 + z_2^5.$$

Let $s_2 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_2(x) = x$, $s_2(y) = y$ and $s_2(z) = z + (\frac{-\alpha_1}{10})^{\frac{1}{2}} y$.

Set $y_1 = \gamma^{\frac{1}{4}} y$ and suppose that δ_1, α_3 are the unit coefficients of $y_1^3 z_2^2$ and $y_1 z_2^4$ then

$$\gamma y^4 + \delta y^3 z_2^2 + \alpha_2 y z_2^4 + z_2^5 = y_1^4 + \delta_1 y_1^3 z_2^2 + \alpha_3 y_1 z_2^4 + z_2^5.$$

Let $s_3 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_3(x) = x$, $s_3(y) = \gamma^{\frac{1}{4}} y$ and $s_3(z) = z$.

Set $y_2 = y_1 + \frac{\delta_1}{4} z_2^2$ and suppose that α_4 and δ_2 are the unit coefficients of $y_2 z_2^4$ and z_2^5 then

$$y_1^4 + \delta_1 y_1^3 z_2^2 + \alpha_3 y_1 z_2^4 + z_2^5 = y_2^4 + \alpha_4 y_2 z_2^4 + \delta_2 z_2^5.$$

Let $s_4 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_4(x) = x$, $s_4(y) = y + \frac{\delta_1}{4} z^2$ and $s_4(z) = z$.

Set $z_3 = \alpha^{\frac{1}{4}} z_2$ and $\beta = \frac{\delta_2}{\alpha_4^{\frac{5}{4}}}$ then we can rewrite

$$y_2^4 + \alpha_4 y_2 z_2^4 + \delta_2 z_2^5 = y_2^4 + y_2 z_3^4 + \beta z_3^5.$$

Let $s_5 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_5(x) = x$, $s_5(y) = y$ and $s_4(z) = \alpha^{\frac{1}{4}} z$.

Let $s : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s = s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_5$. Then s takes $y^4 + yz^4 + \beta z^5$ to a. $\square$

In cases I-V we can determine which i and j force $y^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ and $z^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ but in case VI we need the following lemma to reduce $(y^2 + z^3)^2 + \alpha y z^r + \beta z^m$ to one of the following two forms: $f = (y^2 + z^3)^2 + \gamma z^6$ or $g = (y^2 + z^3)^2 + \gamma y z^r$.

Lemma 3.37 *Let k is an algebraically closed field of characteristic $p > n+1$ and suppose $R = k[[x, y, z]]/(x^2 - a)$ where $a = (y^2 + z^3)^2 + \alpha y z^r + \beta z^{r+m}$, α and β are units and $n \geq 5$. Then for $m = 1, 2$ there exists an automorphism of $k[[x, y, z]]$ taking $(y^2 + z^3)^2 + \gamma z^6$ to a where γ is a unit defined in the proof and for $m \geq 3$ there exists an automorphism of $k[[x, y, z]]$ taking $(y^2 + z^3)^2 + \gamma y z^r$ to a.*

Proof. First assume that $m \geq 3$. Set $y_1 = y + \frac{\beta}{\alpha} z^m$ and suppose that γ_1 is the transformed coefficient of z^3. Then

$$(y^2 + z^3)^2 + \alpha y z^r + \beta z^{r+m} = (y_1^2 + \gamma_1 z^3)^2 + \alpha y_1 z^r.$$

Let $s_1 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_1(x) = x$, $s_1(y) = y + \frac{\beta}{\alpha} z^m$ and $s_1(z) = z$.

Let $z_1 = \gamma_1^{\frac{1}{3}} z$ and suppose that γ is the transformed coefficient of $y_1 z^r$. Then

$$(y_1^2 + \gamma_1 z^3)^2 + \alpha y_1 z^r + \beta z^{r+m} = (y_1^2 + z_1^3)^2 + \gamma y_1 z_1^r.$$

Let $s_2 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_2(x) = x$, $s_2(y) = y$ and $s_2(z) = \gamma_1^{\frac{1}{3}} z$.

Let $s : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s = s_1 \circ s_2$. Then s takes $(y^2 + z_1^3)^2 + \gamma y z^r$ to a.

Now assume $m = 1$. Then $a = (y^2 + z^3)^2 + \alpha y z^r + \beta z^{r+1}$. Let $z_1 = \beta^{\frac{1}{n+1}} z$ and suppose α_1 and γ_1 are the transformed coefficients of $y z_1^r$ and z_1^3 respectively. Then

$$(y^2 + z^3)^2 + \alpha y z^r + \beta z^{r+1} = (y^2 + \gamma_1 z_1^3)^2 + \alpha_1 y z_1^r + z_1^{r+1}.$$

Let $s_1 : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by $s_1(x) = x$, $s_1(y) = y$ and $s_1(z) = \beta^{\frac{1}{n+1}} z$.

We set $z_2 = z_1 + \frac{\alpha_1}{r+1} y$ and expand $(y^2 + \gamma_1 z_1^3)^2$. Suppose α_2, β_2, γ_2 and δ_2 are the transformed coefficients of $y^2 z_2^3$, $y^3 z_2^2$, z_2^6 and y^4 respectively. Then

$$(y^2 + \gamma_1 z_1^3)^2 + \alpha_1 y z_1^r + z_1^{r+1} = \delta_2 y^4 + \beta_2 y^3 z_2^2 + \alpha_2 y^2 z_2^3 + \gamma_2 z_2^6.$$

Let $s_2 : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by $s_2(x) = x$, $s_2(y) = y$ and $s_2(z) = z + \frac{\alpha_1}{r+1} y$.

Let $y_1 = \delta_2^{\frac{1}{4}} y$ and suppose that α_3, β_3 and γ_3 are the transformed coefficients of $y_1^2 z_2^3$, $y_1^3 z_2^2$ and z_2^6 respectively. Then

$$\delta_2 y^4 + \beta_2 y^3 z_2^2 + \alpha_2 y^2 z_2^3 + \gamma_2 z_2^6 = y_1^4 + \beta_3 y_1^3 z_2^2 + \alpha_3 y_1^2 z_2^3 + \gamma_3 z_2^6.$$

Let $s_3 : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by $s_3(x) = x$, $s_3(y) = \delta_2^{\frac{1}{4}} y$ and $s_3(z) = z$.

Let $y_2 = y_1 + \frac{\beta_2}{4} z_2^2$ and suppose that α_4 and γ_4 are the transformed coefficients of z_2^3 and z_2^6 after completing the square then

$$y_1^4 + \beta_3 y_1^3 z_2^2 + \alpha_3 y_1^2 z_2^3 + \gamma_3 z_2^6 = (y_2^2 + \alpha_4 z_2^3)^2 + \gamma_4 z_2^6.$$

Let $s_4 : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by $s_4(x) = x$, $s_4(y) = y + \frac{\beta_2}{4} z^2$ and $s_4(z) = z$.

Let $z_3 = \alpha_4^{\frac{1}{3}} z_2$ and suppose that γ is the transformed coefficients of z_3^6. Then

$$(y_2^2 + \alpha_4 z^3)^2 + \gamma_4 z^6 = (y_2^2 + z_3^3)^2 + \gamma z_3^6.$$

Let $s_5 : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by $s_5(x) = x$, $s_5(y) = y$ and $s_5(z) = \alpha_4^{\frac{1}{3}} z$.

Let $s : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by

$$s = s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_5.$$

Then s takes $(y^2 + z^3)^2 + \gamma z^6$ to a.

Now assume $m = 2$. Then $a = (y^2 + z^3)^2 + \alpha y z^r + \beta z^{r+2}$. We set $y_1 = y + \frac{\beta}{\alpha} z^2$ and expand $(y^2 + z^3)^2$ such that β_1, γ_1, δ_1 and ϵ_1 are the transformed coefficients of $y_1^3 z^2$, $y_1^2 z^3$, $y_1 z^5$ and z^6 respectively. Then

$$(y^2 + z^3)^2 + \alpha y z^r + \beta z^{r+2} = y_1^4 + \beta_1 y_1^3 z^2 + \gamma_1 y_1^2 z^3 + \delta_1 y_1 z^5 + \epsilon_1 z^6.$$

Let $s_1 : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by $s_1(x) = x$, $s_1(y) = y + \frac{\beta}{\alpha} z^2$ and $s_1(z) = z$.

Let $z_1 = \epsilon_1^{\frac{1}{6}} z$ and suppose that β_2, γ_2 and δ_2 are the transformed coefficients of $y_1^3 z_1^2$, $y_1^2 z_1^3$ and $y_1 z_1^5$ respectively. Then

$$y_1^4 + \beta_1 y_1^3 z^2 + \gamma_1 y_1^2 z^3 + \delta_1 y_1 z^5 + \epsilon_1 z^6 = y_1^4 + \beta_2 y_1^3 z_1^2 + \gamma_2 y_1^2 z_1^3 + \delta_2 y_1 z_1^5 + z_1^6.$$

Let $s_2 : k[[x,y,z]] \to k[[x,y,z]]$ be the automorphism defined by $s_2(x) = x$, $s_2(y) = y$ and $s_2(z) = \epsilon_1^{\frac{1}{6}} z$.

Let $z_2 = z_1 + \frac{\delta_2}{6} y_1$ and suppose that β_3, γ_3 and δ_3 are the transformed coefficients of $y_1^3 z_2^2$, $y_1^2 z_2^3$ and y_1^4 respectively. Then

$$y_1^4 + \beta_2 y_1^3 z_1^2 + \gamma_2 y_1^2 z_1^3 + \delta_2 y_1 z_1^5 + z_1^6 = \delta_3 y_1^4 + \beta_3 y_1^3 z_2^2 + \gamma_3 y_1^2 z_2^3 + z_2^6.$$

Let $s_3 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_3(x) = x$, $s_3(y) = y$ and $s_3(z) = z + \frac{\delta_2}{6} y$.

Let $y_2 = \delta_3^{\frac{1}{4}} y_1$ and suppose that β_4, γ_4 and δ_4 are the transformed coefficients of $y_2^3 z_2^2$, $y_2^2 z_2^3$ and z_2^6 respectively. Then

$$\delta_3 y_1^4 + \beta_3 y_1^3 z_2^2 + \gamma_3 y_1^2 z_2^3 + z_2^6 = y_2^4 + \beta_4 y_2^3 z_2^2 + \gamma_4 y_2^2 z_2^4 + \delta_4 z_2^6.$$

Let $s_4 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_4(x) = x$, $s_4(y) = \delta_3^{\frac{1}{4}} y$ and $s_4(z) = z$.

Let $y_3 = y_2 + \frac{\beta_4}{4} z_2^2$ and suppose that γ_5 and δ_5 are the transformed coefficients of z_2^3 and z_2^6 respectively after completing the square. Then

$$y_2^4 + \beta_4 y_2^3 z_2^2 + \gamma_4 y_2^2 z_2^3 + \delta_4 z_2^6 = (y_3^2 + \gamma_5 z_2^3)^2 + \delta_5 z_2^6.$$

Let $s_5 : k[[x, y, z]] \to k[[x, y, z]]$ be the automorphism defined by $s_5(x) = x$, $s_5(y) = y + \frac{\beta_4}{4} z^2$ and $s_5(z) = z$.

Let $z_3 = \gamma_5^{\frac{1}{3}} z_2$ and suppose that γ is the transformed coefficient of z_3^6. Then

$$(y_3^2 + \gamma_5 z_2^3)^2 + \delta_5 z_2^6 = (y_3^2 + z_2^3)^2 + \gamma z_2^6.$$

Let $s_5 : k[[y, z]] \to k[[y, z]]$ be the automorphism defined by $s_5(y) = y$ and $s_5(z) = \gamma_5^{\frac{1}{3}} z$.

Let $s : k[[y, z]] \to k[[y, z]]$ be the automorphism defined by

$$s = s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_5 \circ s_6.$$

Then s takes $(y^2 + z^3)^2 + \gamma z^6$ to a. □

Now we are ready to begin classifying those $a \in (y,z)^4 \backslash (y,z)^5$.

Theorem 3.38 *Suppose the characteristic of k is $p > max(\lambda(R/(\Delta(x^2 - a)),8))$. Let $R = k[[x,y,z]]/(x^2 - a)$ where $a \in (y,z)^4 \backslash (y,z)^5$. Then $\mathfrak{m}$ is the test ideal if and only if after a change of variables we can write a in one of the following forms:*

1) $a = yz(y+z)(y+\lambda z)$, *λ a unit,*

2) $a = (y^2 + \alpha z^r)z(y+z)$, *$\alpha$ a unit and $n \geq 3$,*

3) $a = (y^2 + \alpha z^r)(z^2 + \beta y^m)$, *$\alpha$, β units and $n, m \geq 3$,*

4) $a = z(y^3 + z^i)$, $3 \leq i < 9$,

5) $a = z(y^3 + yz^i)$, $3 \leq i < 6$,

6) $a = z(y^3 + yz^i + z^{i+j})$, $3 \leq i < 6$ *and* $1 \leq j \leq i$, $i = 6$ *and* $1 \leq j \leq 2$ *or* $i = 7$ *and* $j = 1$,

7) $a = y^4 + z^i$, $5 \leq i < 8$,

8) $a = y^4 + yz^i$, $4 \leq i < 6$,

9) $a = y^4 + yz^i + \alpha z^{i+j}$, $4 \leq i < 6$ *and* $1 \leq j < i$,

10) $a = (y^2 + z^3)^2 + \alpha yz^r$, *$n \geq 5$ and α a unit,*

11) $a = (y^2 + z^3)^2 + \alpha z^r$, *$n \geq 6$ and α a unit.*

Proof. If $a \in (y,z)^4 \backslash (y,z)^5$, then the initial form of a will either be:

A) a product of four independent linear terms, $yz(y+z)(y+\lambda z)$,

B) a product of a square and two independent linear terms, $y^2z(y+z)$,

C) a product of two independent squares, y^2z^2,

D) a product of a cube and an independent linear term, y^3z,

E) a fourth power of a linear term, y^4.

Each initial form will give us at least one and in some cases, more than one isomorphism class of rings that have test ideal equal to the maximal ideal. To check each isomorphism class has the test ideal equal to the maximal ideal, by theorem 3.24 we need only check that $a^{\frac{q+1}{2}} \in (y^q, z^q)$ and $a^{\frac{q+1}{2}}y^q \notin (y^{2q}, z^{2q})$ and $a^{\frac{q+1}{2}}z^q \notin (y^{2q}, z^{2q})$. Theorem 3.25 guarantees that $a^{\frac{q+1}{2}} \in (y^q, z^q)$ since $a \in (y,z)^4 \cap k[[y,z]]$.

A) If the initial form is $yz(y+z)(y+\lambda z)$, then by Lemma A.6, $yz(y+z)(y+\lambda z)+b$ where $b \in (y,z)^5$ can be rewritten after some change of variables as $yz(y+z)(y+\mu z)$ where μ is a unit in $k[[y,z]]$. Assume μ is in k for the moment. Note that $x^2 - a = x^2 - yz(y+z)(y+\mu z)$ is a quasihomogeneous polynomial with $\deg(x) = 2$ and $\deg(y) = \deg(z) = 1$. By Corollary 3.23, $\tau = \mathfrak{m}$ if and only if $\deg(x) = 2 \geq 1 + 1 = \deg(y) + \deg(z)$, $\deg(xy) = 3 < 2 + 2 = \deg(y^2) + \deg(z^2)$ and $\deg(xz) = 3 < 2 + 2 = \deg(y^2) + \deg(z^2)$.

For $\mu \notin k$, by Remark 3.24 we need to show

$$y^q(yz(y+z)(y+\mu z))^{\frac{q+1}{2}} \text{ and } z^q(yz(y+z)(y+\mu z))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence, we need only show there exists a monomial in the expansion of

$$y^q(yz(y+z)(y+\mu z))^{\frac{q+1}{2}} \text{ and } z^q(yz(y+z)(y+\mu z))^{\frac{q+1}{2}}$$

that is not contained in (y^{2q}, z^{2q}). Notice we can write

$$\mu = \Sigma_{i,j\geq 0}\mu_{ij}y^i z^j$$

where $\mu_{ij} \in k$. We know from above that there exists monomials with nonzero coefficient modulo p in the expansion of

$$y^q(yz(y+z)(y+\mu_{00}z))^{\frac{q+1}{2}} \text{ and } z^q(yz(y+z)(y+\mu_{00}z))^{\frac{q+1}{2}}$$

that are not contained in (y^{2q}, z^{2q}) and these monomials have degree $3q+2$. If we multiply any monomial by $y^i z^j$ this will only give us a monomial of a larger degree. Thus these monomials will also be in the expansion of

$$y^q(yz(y+z)(y+\mu z))^{\frac{q+1}{2}} \text{ and } z^q(yz(y+z)(y+\mu z))^{\frac{q+1}{2}}.$$

Hence,

$$y^q(yz(y+z)(y+\mu z))^{\frac{q+1}{2}} \text{ and } z^q(yz(y+z)(y+\mu z))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

This gives us case 1).

B) If the initial form is $y^2z(y+z)$, then by Lemma A.7, $y^2z(y+z)+b$ where $b \in (y,z)^5$ can be rewritten after some change of variables as $(y^2+\alpha z^r)z(y+z)$ where α is a unit in $k[[y,z]]$ and $n \geq 3$. Assume α is in k for the moment. To show that

$$y^q((y^2+\alpha z^r)z(y+z))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

we need only exhibit that there exists a monomial in the expansion of

$$y^q((y^2+\alpha z^r)z(y+z))^{\frac{q+1}{2}}$$

that is not contained in (y^{2q}, z^{2q}). The monomials look like

$$y^{\frac{5q+3}{2}-2r-s} z^{\frac{q+1}{2}+rr+s},$$

for $0 \leq r, s \leq \frac{q+1}{2}$. Taking $r = 1$ and $s = \frac{q+1}{2}$ gives us the monomial

$$y^{2q-1} z^{q+1+r}$$

which is not in (y^{2q}, z^{2q}) for $q > n+1$. Notice if α were not contained in k, then

$$\alpha = \Sigma_{i,j \geq 0} \alpha_{ij} y^i z^j.$$

Suppose that some $y^i z^j$ in the expansion of α multiplied by another monomial in the expansion of

$$y^q((y^2 + \alpha_{00} z^r) z(y+z))^{\frac{q+1}{2}}$$

contributes another monomial in the form

$$y^{2q-1} z^{q+1+r}.$$

Then for some r and s we have the following system of equations

$$\frac{5q+3}{2} + (i-2)r - s = q - 1$$

and

$$\frac{q+1}{2} + (n+j)r + s = q + 1 + n.$$

Solving this system we see that

$$r = \frac{n-2}{n-2+i+j} \text{ and } s = \frac{q+5}{2} + (i-2)(\frac{n-2}{n-2+i+j}),$$

neither of which are integers unless $i = j = 0$. Thus no terms in the expansion of α will contribute any more monomials of the form

$$y^{2q-1}z^{q+1+r}.$$

Thus

$$y^q((y^2+\alpha z^r)z(y+z))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

To show

$$z^q((y^2+\alpha z^r)z(y+z))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}),$$

we note that

$$y^{\frac{3(q+1)}{2}}z^{\frac{3q+1}{2}}$$

is a monomial in the expansion of

$$z^q((y^2+\alpha z^r)z(y+z))^{\frac{q+1}{2}}$$

and

$$y^{\frac{3(q+1)}{2}}z^{\frac{3q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence, by Remark 3.24 we have case 2).

C) If the initial form is y^2z^2, then by Lemma A.8, y^2z^2+b where $b \in (y,z)^5$ can be rewritten after some change of variables as $(y^2+\alpha z^r)(z^2+\beta y^m)$ where α and β are units in $k[[y,z]]$ and $n, m \geq 3$. Assume α and β are in k for the moment. To show that

$$y^q((y^2+\alpha z^r)(z^2+\beta y^m))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

we need only show there exists a monomial in the expansion of

$$y^q((y^2+\alpha z^r)(z^2+\beta y^m))^{\frac{q+1}{2}}$$

that is not contained in (y^{2q}, z^{2q}). The monomials look like

$$y^{(2q+1)-2r+ms}z^{(q+1)-2s+rr},$$

for $0 \leq r, s \leq \frac{q+1}{2}$. Taking $r = 1$ and $s = 0$ gives us the monomial

$$y^{2q-1}z^{q+1+r}$$

which is not in (y^{2q}, z^{2q}) for $q > n+1$. Notice if α or β were not contained in k

$$\alpha = \Sigma_{i,j\geq 0}\alpha_{ij}y^iz^j \text{ and } \beta = \Sigma_{i,j\geq 0}\beta_{ij}y^iz^j.$$

Suppose that some y^iz^j in the expansion of α and y^kz^l in the expansion of β multiplied by another monomial in the expansion of

$$y^q((y^2+\alpha_{00}z^r)(z^2+\beta_{00}y^m))^{\frac{q+1}{2}}$$

contributes another monomial in the form

$$y^{2q-1}z^{q+1+r}.$$

Then for some r and s we have the following system of equations

$$(i-2)r+(m+k)s = -2 \text{ and } (n+j)r+(l-2)s = n.$$

Since $n, m \geq 3$ then $(n(k+m)+2(l-2)) > 0$ and by assumption $i, j, r, s \geq 0$, thus $s = 0$. Plugging $s = 0$ into the above two equations yields $i = j = 0$ and $r = 1$.

Thus no terms in the expression of α or β will contribute any more monomials of the form

$$y^{2q-1}z^{q+1+r}.$$

Thus

$$y^q((y^2+\alpha z^r)(z^2+\beta y^m)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

By symmetry we show

$$z^q(y^2+\alpha z^r)(z^2+\beta y^m)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence by Remark 3.24 we have case 3).

D) If the initial form is y^3z, then by Lemma A.9 y^3z+b where $b \in (y,z)^5$ can be rewritten after some change of variables as $z(y^3+az^3)$ where $a \in (y,z)\cap k[[y,z]]$. By the Weierstrass Preparation theorem y^3+az^3 can be replaced by $u(y,z)(y^3+a(z)y+b(z))$ where $u(y,z)$ is a unit and $a(z), b(z) \in k[[z]]$. To determine the distinct isomorphism classes which stem from this form we break this situation down into the three cases: I) $a(z)=0$, II) $b(z)=0$ and III) $a(z)\neq 0$ and $b(z)\neq 0$. Or in otherwords,

I) $z(y^3+\alpha z^i)$, where α is a unit in $k[[z]]$,

II) $z(y^3+\alpha yz^i)$, where α is a unit in $k[[z]]$,

III) $z(y^3+\alpha yz^i+\beta z^j)$, where α and β are units in $k[[z]]$.

D.I) As long as $\alpha \in k$, $x^2-a = x^2-z(y^3+\alpha z^i)$ is a quasihomogeneous polynomial with $\deg(x)=3i+3$, $\deg(y)=2i$ and $\deg(z)=6$. By Corollary 3.23 $\tau = \mathrm{m}$ if and only if $\deg(x)=3i+3\geq 2i+6=\deg(y)+\deg(z)$, $\deg(xy)=5i+3<$

$4i+12 = \deg(y^2)+\deg(z^2)$ and $\deg(xz) = 3i+9 < 4i+12 = \deg(y^2)+\deg(z^2)$. In otherwords, $\tau = \mathfrak{m}$ if and only if $3 \leq i < 9$. Note that when $i = 3$, $a = z(y^3+\alpha z^3)$ is a product of four independent linear factors. In case A above we have shown that such a force $k[[x,y,z]]/(x^2-a)$ to have test ideal equal to the maximal ideal.

When $\alpha \notin k$, by Remark 3.24, we need only check that

$$y^q(z(y^3+\alpha z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3+\alpha z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

as long as $4 \leq i < 9$ by looking at the monomials in the expansion of both expressions above.

We can write

$$\alpha = \Sigma_{k,l\geq 0}\alpha_{kl}y^k z^l$$

where $\alpha_{kl} \in k$. The monomials will be of the form

$$y^{\frac{5q+3}{2}+(k-3)r}z^{\frac{q+1}{2}+(i+l)r} \text{ and } y^{\frac{3(q+1)}{2}+(k-3)r}z^{\frac{3q+1}{2}+(i+l)r},$$

for some $0 \leq r \leq \frac{q+1}{2}$. Since $\tau = \mathfrak{m}$ for $k[[x,y,z]]/(x^2-(z(y^3+\alpha_{00}z^i)))$ then we know there must be some monomial with nonzero coefficient modulo p in the expansion of

$$y^q(z(y^3+\alpha_{00}z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3+\alpha_{00}z^i))^{\frac{q+1}{2}}$$

which is not contained in (y^{2q}, z^{2q}). Assume these monomials are such:

$$y^{\frac{5q+3}{2}-3r}z^{\frac{q+1}{2}+ir} \text{ and } y^{\frac{3(q+1)}{2}-3r}z^{\frac{3q+1}{2}+ir}.$$

Suppose α contributes other monomials with the same exponents for both y

and z. Then there exists an $s \neq r$ such that

$$y^{\frac{5q+3}{2}-3r} z^{\frac{q+1}{2}+ir} = y^{\frac{5q+3}{2}+(k-3)s} z^{\frac{q+1}{2}+(i+l)s}$$

and

$$y^{\frac{3(q+1)}{2}-3r} z^{\frac{3q+1}{2}+ir} = y^{\frac{3(q+1)}{2}+(k-3)s} z^{\frac{3q+1}{2}+(i+l)s}$$

From both equalities we see that

$$-3r = (k-3)s \text{ and } ir = (i+l)s.$$

In both cases $s = r$ and

$$y^q(z(y^3+\alpha z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3+\alpha z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Now by Lemma 3.31 there exists an automorphism from $z(y^3+\alpha z^i)$ to $z(y^3+z^i)$. Thus $k[[x,y,z]]/(x^2 - z(y^3+z^i))$ has test ideal equal to the maximal ideal for $4 \leq i < 9$. This gives us 4).

D.II) As long as $\alpha \in k$, $x^2 - a = x^2 - z(y^3 + \alpha y z^i)$ is a quasihomogeneous polynomial with $\deg(x) = 3i+2$, $\deg(y) = 2i$ and $\deg(z) = 4$. By Corollary 3.23 $\tau = \mathfrak{m}$ if and only if $\deg(x) = 3i+2 \geq 2i+4 = \deg(y)+\deg(z)$, $\deg(xy) = 5i+2 < 4i+8 = \deg(y^2)+\deg(z^2)$ and $\deg(xz) = 3i+6 < 4i+8 = \deg(y^2)+\deg(z^2)$. In otherwords, $\tau = \mathfrak{m}$ if and only if $2 \leq i < 6$. Note that when $i = 2$, $a = z(y^3+\alpha y z^2)$ is a product of four independent linear factors. In case A above we have shown that such a force $k[[x,y,z]]/(x^2-a)$ to have test ideal equal to the maximal ideal.

For $\alpha \notin k$, by Remark 3.24 we need to check that

$$y^q(z(y^3+\alpha y z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3+\alpha y z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

as long as $3 \leq i < 6$. To do this we look at the monomials in the expansion of both expressions above. Since we can write

$$\alpha = \Sigma_{k,l \geq 0} \alpha_{kl} y^k z^l,$$

the monomials in the expansion of

$$y^q(z(y^3 + \alpha y z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3 + \alpha y z^i))^{\frac{q+1}{2}}$$

are of the form

$$y^{\frac{5q+3}{2}+(k-2)r} z^{\frac{q+1}{2}+(i+l)r} \text{ and } y^{\frac{3(q+1)}{2}+(k-2)r} z^{\frac{3q+1}{2}+(i+l)r},$$

for some $0 \leq r \leq \frac{q+1}{2}$. Since $\tau = \mathfrak{m}$ for $k[[x, y, z]]/(x^2 - (z(y^3 + \alpha_{00} y z^i)))$ then we know there must be some monomial with nonzero coefficient modulo p in the expansion of

$$y^q(z(y^3 + \alpha_{00} y z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3 + \alpha_{00} y z^i))^{\frac{q+1}{2}}$$

which is not contained in (y^{2q}, z^{2q}). Assume these monomials are such:

$$y^{\frac{5q+3}{2}-2r} z^{\frac{q+1}{2}+ir} \text{ and } y^{\frac{3(q+1)}{2}-2r} z^{\frac{3q+1}{2}+ir}.$$

Suppose α contributes other monomials with the same exponents for both y and z. Then there exists an $s \neq r$ such that

$$y^{\frac{5q+3}{2}-2r} z^{\frac{q+1}{2}+ir} = y^{\frac{5q+3}{2}+(k-2)s} z^{\frac{q+1}{2}+(i+l)s}$$

or

$$y^{\frac{3(q+1)}{2}-2r} z^{\frac{3q+1}{2}+ir} = y^{\frac{3(q+1)}{2}+(k-2)s} z^{\frac{3q+1}{2}+(i+l)s}$$

From both equalities we see that

$$-2r = (k-2)s \text{ and } ir = (i+l)s.$$

In both cases $s = r$ and

$$y^q(z(y^3+\alpha yz^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3+\alpha yz^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Now by Lemma 3.31 there exists an automorphism from $z(y^3+\alpha yz^i)$ to $z(y^3+yz^i)$. Thus $k[[x,y,z]]/(x^2-z(y^3+yz^i))$ have test ideal equal to the maximal ideal for $3 \leq i < 6$. This gives us 5).

D.III) We need only consider the case when $i < j$ since Lemma 3.31 implies that if $i \geq j$ then there exists an automorphism of $k[[x,y,z]]$ taking $x^2-z(y^3+\alpha yz^i+\beta z^j)$ to $(x^2-z(y^3+z^j))$ which was case 1) for $4 \leq j < 9$.

For simplicity replace j by $i+j$ where $j \geq 1$. If $j \geq i$, Lemma 3.33 implies that

$$k[[x,y,z]]/(x^2-z(y^3+\alpha yz^i+\beta z^{i+j})) \cong k[[x,y,z]]/(x^2-z(y^3+yz^i))$$

which is case 5) for $3 \leq i < 6$. Thus we need only consider when

$$a = z(y^3+\alpha yz^i+\beta z^{i+j})$$

for $3 \leq i < 6$ and $1 \leq j < i$, $i = 6, 1 \leq j \leq 2$ or $i = 7, j = 1$.

By Remark 3.24 we show that the test ideal must be the maximal ideal by looking at the monomials in the expansion of

$$y^q(z(y^3+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}}$$

and

$$z^q(z(y^3+\alpha y z^i+\beta z^{i+j}))^{\frac{q+1}{2}}.$$

Taking into account that

$$\alpha=\Sigma_{k,l\geq 0}\alpha_{kl}y^k z^l \text{ and } \beta=\Sigma_{m,r\geq 0}\beta_{mr}y^m z^r,$$

these monomials are in the form

$$y^{\frac{5q+3}{2}+(m-3)r+(k-2)s}z^{\frac{q+1}{2}+(i+j+r)r+(i+l)s}$$

and

$$y^{\frac{3(q+1)}{2}+(m-3)r+(k-2)s}z^{\frac{3(q+1)}{2}+(i+j+r)r+(i+l)s}.$$

Setting $r=0$ in the first equality gives us monomials in the expansion of

$$y^q(z(y^3+\alpha y z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3+\alpha y z^i))^{\frac{q+1}{2}}$$

and we know from D.II, above that $xy, xz \notin (y^2,z^2)^*$ for $3\leq i<6$ and in particular

$$y^{\frac{5q+3}{2}-2s}z^{\frac{q+1}{2}+is} \text{ or } y^{\frac{3(q+1)}{2}-2s}z^{\frac{3q+1}{2}+is}\notin(y^{2q},z^{2q}).$$

As long as there is no other monomial with the same exponents for both y and z then

$$y^q(z(y^3+\alpha y z^i+\beta z^{i+j}))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3+\alpha y z^i+\beta z^{i+j}))^{\frac{q+1}{2}}\notin(y^{2q},z^{2q}).$$

Suppose not, then for some pair (t,u),

$$y^{\frac{5q+3}{2}-2s}z^{\frac{q+1}{2}+is}=y^{\frac{5q+3}{2}+(m-3)t+(k-2)u}z^{\frac{q+1}{2}+(i+j+r)t+(i+l)u}$$

or

$$y^{\frac{3(q+1)}{2}-2s} z^{\frac{3q+1}{2}+is} = y^{\frac{3(q+1)}{2}+(m-3)t+(k-2)u} z^{\frac{3(q+1)}{2}+(i+j+r)t+(i+l)u}.$$

Both of the above equalities imply

$$-2s = (m-3)t + (k-2)u \text{ and } is = (i+j+n)t + (i+l)u.$$

As long as $j > \frac{i}{2}$, $2(j+n) + (m-1)i > 0$. Thus for $3 \leq i < 6$ and $2j > i$, $t = 0$ and $u = s$.

Setting $s = 0$ in the first equality gives us monomials in the expansion of

$$y^q(z(y^3 + \beta z^{i+j}))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3 + \alpha y z^i))^{\frac{q+1}{2}}$$

and we know from D.I, above that $xy, xz \notin (y^2, z^2)^*$ for $5 \leq i+j < 8$ and in particular

$$y^{\frac{5q+3}{2}-3r} z^{\frac{q+1}{2}+(i+j)r} \text{ or } y^{\frac{3(q+1)}{2}-3r} z^{\frac{3q+1}{2}+(i+j)r} \notin (y^{2q}, z^{2q}).$$

As above we need only show that there exists no other monomial with the same exponents for both y and z.

Suppose not, then for some pair (t, u),

$$y^{\frac{5q+3}{2}-3r} z^{\frac{q+1}{2}+(i+j)r} = y^{\frac{5q+3}{2}+(m-3)t+(k-2)u} z^{\frac{q+1}{2}+(i+j+r)t+(i+l)u}$$

or

$$y^{\frac{3(q+1)}{2}-3r} z^{\frac{3q+1}{2}+(i+j)r} = y^{\frac{3(q+1)}{2}+(m-3)t+(k-2)u} z^{\frac{3(q+1)}{2}+(i+j+r)t+(i+l)u}.$$

Both of the above equalities imply

$$-3r = (m-3)t + (k-2)u \text{ and } (i+j)r = (i+j+n)t + (i+l)u.$$

As long as $j < \frac{i}{2}$, $(k+1)i + (k-2)j + 3l > 0$. Thus for $3 \leq i < 6$ and $2j < i$, $t = r$ and $u = 0$.

The only case that we haven't accounted for is when $i = 2j$ or $i = 4$ and $j = 2$. But if $\alpha, \beta \in k$, $x^2 - z(y^3 + \alpha y z^4 + \beta z^6)$ is a quasihomogeneous polynomial with the following degrees: $\deg(x) = 7, \deg(y) = 4$, and $\deg(z) = 2$. By Corollary 3.23, $r = \mathrm{m}$ if and only if $\deg(x) = 7 \geq 4+2 = \deg(y) + \deg(z)$, $\deg(xy) = 11 < 8+4 = \deg(y) + \deg(z)$ and $\deg(xz) = 9 < 8+4 = \deg(y) + \deg(z)$.

When $\alpha, \beta \notin k$, we note that as above there exists some r such that

$$y^{\frac{5q+3}{2}-3r} z^{\frac{q+1}{2}+6r} \text{ or } y^{\frac{3(q+1)}{2}-3r} z^{\frac{3q+1}{2}+6r} \notin (y^{2q}, z^{2q})$$

and the sum of the coefficients of this monomial in

$$y^q(z(y^3 + \alpha_{00} y z^4 + \beta_{00} z^6))^{\frac{q+1}{2}} \text{ or } z^q(z(y^3 + \alpha_{00} y z^4 + \beta_{00} z^6))^{\frac{q+1}{2}}$$

are nonzero modulo p. We have shown above that if α and β contributed another monomial of the above form then the following equalities would arise:

$$y^{\frac{5q+3}{2}-3r} z^{\frac{q+1}{2}+6r} = y^{\frac{5q+3}{2}+(m-3)t+(k-2)u} z^{\frac{q+1}{2}+(6+r)t+(4+l)u}$$

or

$$y^{\frac{3(q+1)}{2}-3r} z^{\frac{3q+1}{2}+6r} = y^{\frac{3(q+1)}{2}+(m-3)t+(k-2)u} z^{\frac{3(q+1)}{2}+(6+r)t+(4+l)u};$$

both implying

$$0 = (4m+3n)t + ((k+1)4 + (k-2)2 + 3l)u = (4m+3n)t + (6k+3l)u = 0.$$

Since $u, t, k, l, m, n \geq 0$, if $t \neq 0$ then $m = n = 0$ and if $u \neq 0$ then $k = l = 0$. Since by assumption k, l, m, n cannot be zero simultaneously then either $t = 0$ or $u = 0$. If $u = 0$ then $t = r$. If $t = 0$, then $2u = 3r$. We need only choose r to be odd to get a contradiction. Thus $t = r$ and $u = 0$.

By Lemma 3.31, there exists an automorphism taking $x^2 - (y^3 + \alpha y z^i + \beta z^{i+j})$ to $x^2 - (y^3 + yz^i + \gamma z^{i+j})$ for a unit γ. Thus $k[[x, y, z]]/(x^2 - z(y^3 + yz^i + \gamma z^{i+j}))$ has test ideal equal to the maximal ideal for $3 \leq i < 6$ and $2 \leq j < i$ and $i = 6$ and $j = 2$. This gives case 6).

E) If the initial form is y^4, then by Lemma A.10 $y^4 + b$ where $b \in (y, z)^5$ can be rewritten after some change of variables as $(y^2 + cz^i)^2 + dz^4$ where c is either equal to 0 or 1 and $d \in (y, z)$. Suppose first that $c = 0$. Since $d \in (y, z)$ it must be in the form $a(y, z)y + b(y, z)z$. To analyze the isomorphism classes generated from this form we must look at the following three cases:

I) $y^4 + \alpha z^i$, where α is a unit in $k[[y, z]]$,

II) $y^4 + \alpha y z^i$, where α is a unit in $k[[y, z]]$,

III) $y^4 + \alpha y z^i + \beta z^j$, where α and β are units in $k[[y, z]]$.

E.I) As long as $\alpha \in k$, $x^2 - a = x^2 - (y^4 + \alpha z^i)$ is a quasihomogeneous polynomial with $\deg(x) = 2i$, $\deg(y) = i$ and $\deg(z) = 4$. By Corollary 3.23 $\tau = \mathfrak{m}$ if and only if $\deg(x) = 2i \geq i + 4 = \deg(y) + \deg(z)$, $\deg(xy) = 3i < 2i + 8 = \deg(y^2) + \deg(z^2)$ and $\deg(xz) = 2i + 4 < 2i + 8 = \deg(y^2) + \deg(z^2)$. In otherwords, $\tau = \mathfrak{m}$ if and only if $4 \leq i < 8$. Note that when $i = 4$, $a = y^4 + \alpha z^4$ is a product of four independent linear factors. In case A above we have shown that such a force $k[[x, y, z]]/(x^2 - a)$

to have test ideal equal to the maximal ideal.

When $\alpha \notin k$, by Remark 3.24 we need only check that

$$y^q(y^4+\alpha z^i)^{\frac{q+1}{2}} \text{ and } z^q(y^4+\alpha z^i)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

as long as $5 \leq i < 8$ by looking at the monomials in the expansion of both expressions above.

We can write

$$\alpha = \Sigma_{k,l \geq 0} \alpha_{kl} y^k z^l$$

where $\alpha_{kl} \in k$. The monomials will be of the form

$$y^{3q+2+(k-4)r} z^{(i+l)r} \text{ and } y^{2(q+1)+(k-4)r} z^{q+(i+l)r},$$

for some $0 \leq r \leq \frac{q+1}{2}$. Since $\tau = \mathfrak{m}$ for $k[[x,y,z]]/(x^2-(y^4+\alpha_{00}z^i))$ then we know there must be some monomial with nonzero coefficient modulo p in the expansion of

$$y^q(z(y^3+\alpha_{00}z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3+\alpha_{00}z^i))^{\frac{q+1}{2}}$$

which is not contained in (y^{2q}, z^{2q}). Assume these monomials are such:

$$y^{3q+2-4r} z^{ir} \text{ and } y^{2(q+1)-4r} z^{q+ir}.$$

Suppose α contributes other monomials with the same exponents for both y and z. Then there exists an $s \neq r$ such that

$$y^{3q+2-4r} z^{ir} = y^{3q+2+(k-4)s} z^{(i+l)s} \text{ or } y^{2(q+1)-4r} z^{q+ir} = y^{2(q+1)+(k-4)s} z^{q+(i+l)s}$$

Both equalities yield the following two equations:

$$-4 = (k-4)s \text{ and } ir = (i+l)s.$$

In both cases $s = r$ and

$$y^q(y^4 + \alpha z^i)^{\frac{q+1}{2}} \text{ and } y^q(y^4 + \alpha z^i)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

By Lemma 3.35 there exists an automorphism taking $y^4 + \alpha z^i$ to $y^4 + z^i$. Thus $k[[x,y,z]]/(x^2 - z(y^4 + z^i))$ have test ideal equal to the maximal ideal for $5 \leq i < 8$. Thus we have 7).

E.II) As long as $\alpha \in k$, $x^2 - a = x^2 - (y^4 + \alpha y z^i)$ is a quasihomogeneous polynomial with $\deg(x) = 2i$, $\deg(y) = i$ and $\deg(z) = 3$. By Corollary 3.23 $\tau = \mathfrak{m}$ if and only if $\deg(x) = 2i \geq i + 3 = \deg(y) + \deg(z)$, $\deg(xy) = 3i < 2i + 6 = \deg(y^2) + \deg(z^2)$ and $\deg(xz) = 2i + 3 < 2i + 6 = \deg(y^2) + \deg(z^2)$. In otherwords, $\tau = \mathfrak{m}$ if and only if $3 \leq i < 6$. Note that when $i = 3$, $a = y^4 + \alpha y z^3$ is a product of four independent linear factors. In case A above we have shown that such a force $k[[x,y,z]]/(x^2 - a)$ to have test ideal equal to the maximal ideal.

When $\alpha \notin k$, by Remark 3.24 we need only check that

$$y^q(y^4 + \alpha y z^i)^{\frac{q+1}{2}} \text{ and } z^q(y^4 + \alpha y z^i)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

as long as $4 \leq i < 6$ by looking at the monomials in the expansion of both expressions above.

We can write

$$\alpha = \Sigma_{k,l \geq 0} \alpha_{kl} y^k z^l$$

where $\alpha_{kl} \in k$. The monomials will be of the form

$$y^{3q+2+(k-3)r} z^{(i+l)r} \text{ and } y^{2(q+1)+(k-3)r} z^{q+(i+l)r},$$

for some $0 \leq r \leq \frac{q+1}{2}$. Since $\tau = \mathfrak{m}$ for $k[[x,y,z]]/(x^2-(y^4+\alpha_{00}yz^i))$ then we know there must be some monomial with nonzero coefficient modulo p in the expansion of

$$y^q(z(y^3+\alpha_{00}yz^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^3+\alpha_{00}yz^i))^{\frac{q+1}{2}}$$

which is not contained in (y^{2q}, z^{2q}). Assume these monomials are such:

$$y^{3q+2-3r} z^{ir} \text{ and } y^{2(q+1)-3r} z^{q+ir}.$$

Suppose α contributes other monomials with the same exponents for both y and z. Then there exists an $s \neq r$ such that

$$y^{3q+2-3r} z^{ir} = y^{3q+2+(k-3)s} z^{(i+l)s} \text{ or } y^{2(q+1)-3r} z^{q+ir} = y^{2(q+1)+(k-3)s} z^{q+(i+l)s}$$

Both equalities yield the following two equations:

$$-3 = (k-3)s \text{ and } ir = (i+l)s.$$

In both cases $s = r$ and

$$y^q(y^4+\alpha yz^i)^{\frac{q+1}{2}} \text{ and } z^q(y^4+\alpha yz^i)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

By Lemma 3.35 there exists an automorphism taking $y^4+\alpha yz^i$ to y^4+yz^i. Thus $k[[x,y,z]]/(x^2-z(y^4+yz^i))$ have test ideal equal to the maximal ideal for $4 \leq i < 6$. Thus we have 8).

E.III) We need only consider the case when $i < j$ since Lemma 3.31 implies that if $i \geq j$ then there exists an automorphism of $k[[x,y,z]]$ taking $x^2-(y^4+\alpha yz^i+\beta z^j)$ to $(x^2-(y^4+z^j))$ which was case E.I for $5 \leq j < 8$.

For simplicity replace j by $i+j$ where $j \geq 1$. If $j \geq i$, Lemma 3.35 implies that

$$k[[x,y,z]]/(x^2-(y^4+\alpha yz^i+\beta z^{i+j})) \cong k[[x,y,z]]/(x^2-(y^4+yz^i))$$

which is case E.II for $4 \leq i < 6$. Thus we need only consider when

$$a = y^4+\alpha yz^i+\beta z^{i+j}$$

for $4 \leq i < 6$ and $1 \leq j < i$, $i=6, j=1$.

By Remark 3.24 we show that the test ideal must be the maximal ideal by looking at the monomials in the expansion of

$$y^q(y^4+\alpha yz^i+\beta z^{i+j})^{\frac{q+1}{2}}$$

and

$$z^q(y^4+\alpha yz^i+\beta z^{i+j})^{\frac{q+1}{2}}.$$

Taking into account that

$$\alpha = \Sigma_{k,l \geq 0}\alpha_{kl}y^k z^l \text{ and } \beta = \Sigma_{m,r \geq 0}\beta_{mr}y^m z^r,$$

these monomials are in the form

$$y^{3q+2+(m-4)r+(k-3)s}z^{(i+j+r)r+(i+l)s} \text{ and } y^{2(q+1)+(m-4)r+(k-3)s}z^{q+(i+j+r)r+(i+l)s},$$

for $0 \leq r \leq \frac{q+1}{2}$ and $0 \leq s \leq \frac{q+1}{2}-r$.

Setting $r = 0$ in the first equality gives us monomials in the expansion of

$$y^q(y^4 + \alpha y z^i)^{\frac{q+1}{2}} \text{ and } z^q(y^4 + \alpha y z^i)^{\frac{q+1}{2}}$$

and we know from E.II, above that $xy, xz \notin (y^2, z^2)^*$ for $4 \le i < 6$ and in particular

$$y^{3q+2-3s}z^{is} \notin (y^{2q}, z^{2q}) \text{ or } y^{2(q+1)-2s}z^{q+is} \notin (y^{2q}, z^{2q}).$$

As long as there is no other monomial with the same exponents for both y and z then

$$y^q(y^4 + \alpha y z^i + \beta z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^4 + \alpha y z^i + \beta z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Suppose not, then for some pair (t, u),

$$y^{3q+2-3s}z^{is} = y^{3q+2+(m-4)t+(k-3)u}z^{(i+j+r)t+(i+l)u}$$

or

$$y^{2(q+1)-3s}z^{q+is} = y^{2(q+1)+(m-4)t+(k-3)u}z^{q+(i+j+r)t+(i+l)u}.$$

Both of the above equalities imply

$$-3s = (m-4)t + (k-3)u \text{ and } is = (i+j+n)t + (i+l)u.$$

As long as $3j > i$, $3(j+n) + (m-1)i > 0$. Thus for $3 \le i < 6$ and $j \ge 2$, $t = 0$ and $u = s$.

Setting $s = 0$ in the first equality gives us monomials in the expansion of

$$y^q(y^4 + \beta z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^4 + \beta z^{i+j})^{\frac{q+1}{2}}$$

and we know from E.I. above that $xy, xz \notin (y^2, z^2)^*$ for $4 \leq i+j < 8$ and in particular

$$y^{3q+2-4r} z^{(i+j)r} \text{ or } y^{2(q+1)-4r} z^{q+(i+j)r} \notin (y^{2q}, z^{2q}).$$

As above we need only show that there exists no other monomial with the same exponents for both y and z.

Suppose not, then for some pair (t, u),

$$y^{3q+2-4r} z^{(i+j)r} = y^{3q+2+(m-4)t+(k-3)u} z^{(i+j+r)t+(i+l)u}$$

or

$$y^{2(q+1)-4r} z^{q+(i+j)r} = y^{2(q+1)+(m-4)t+(k-3)u} z^{q+(i+j+r)t+(i+l)u}.$$

Both of the above equalities imply

$$-4r = (m-4)t + (k-3)u \text{ and } (i+j)r = (i+j+n)t + (i+l)u.$$

As long as $3j < i$, $(k+1)i + (k-3)j + 4l > 0$. Thus for $3 \leq i < 8$ and $j = 1$, $t = r$ and $u = 0$.

By Lemma 3.35 there exists an automorphism taking $y^4 + \alpha y z^i + \beta z^{i+j}$ to $y^4 + yz^i + \gamma z^{i+j}$ for some unit γ. Thus $k[[x, y, z]]/(x^2 - (y^4 + yz^i + \gamma z^{i+j}))$ has test ideal equal to the maximal ideal for $4 \leq i < 6$ and $j \geq 1$ and for $i = 6$ and $j = 1$. Thus we have 9).

Now consider $(y^2 + z^i)^2 + dz^4$ where $d \in (y, z)$. Since $d \in (y, z)$ it must be in the form $a(z)y + b(z)z$. To analyze the isomorphism classes generated from this form we must look at the following three cases where $n > i$:

IV) $(y^2 + z^i)^2 + \alpha z^r$, where α is a unit in $k[[z]]$,

V) $(y^2 + z^i)^2 + \alpha y z^r$, where α is a unit in $k[[z]]$,

VI) $(y^2 + z^i)^2 + \alpha y z^r + \beta z^m$, where α and β are units in $k[[z]]$.

E.IV) By Remark 3.24 we need to show that

$$y^q((y^2 + z^i)^2 + \alpha z^r)^{\frac{q+1}{2}} \text{ and } z^q((y^2 + z^i)^2 + \alpha z^r)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $i = 3$. To do this we look at the monomials in the expansion of

$$y^q((y^2 + z^i)^2 + \alpha z^r)^{\frac{q+1}{2}} \text{ and } z^q((y^2 + z^i)^2 + \alpha z^r)^{\frac{q+1}{2}}.$$

As long as $\alpha \in k$ the monomials in the expansion of

$$y^q((y^2 + z^i)^2 + \alpha z^r)^{\frac{q+1}{2}} \text{ and } z^q((y^2 + z^i)^2 + \alpha z^r)^{\frac{q+1}{2}}$$

have the form

$$y^{3q+2-4r-2s} z^{is+rr} \text{ and } y^{2(q+1)-4r-2s} z^{q+is+rr}$$

for $0 \leq r \leq \frac{q+1}{2}$ and $0 \leq s \leq q + 1 - 2r$.

Setting $r = 1$ and $s = \frac{q+1}{2}$ in the first expresion above, the monomial we obtain is

$$y^{2q-3} z^{\frac{i(q+1)}{2}+r} \notin (y^{2q}, z^{2q})$$

for $i = 3$ and as long as $q > 2n + 3$ but it is always contained in (y^{2q}, z^{2q}) for $i \geq 4$. Note by Lemma 3.29 the coefficient of $\binom{q-1}{\frac{q+1}{2}} \not\equiv 0 \bmod p$. Thus as long as

$$y^{2q-3} z^{\frac{i(q+1)}{2}+r}$$

is not contained in (y^{2q}, z^{2q}) then

$$((y^2 + z^i)^2 + \alpha z^r)^{\frac{q+1}{2}}$$

is not contained in (y^{2q}, z^{2q}). Setting $r = 1$ and $s = 0$ in the second expression above, the monomial we obtain is

$$y^{2q-2} z^{q+r}$$

which is not contained in (y^{2q}, z^{2q}) for $q > n$.

Note that the monomials are uniquely determined by r and s. Suppose for some t and u

$$y^{3q+2-4r-2s} z^{r\,r+3s} = y^{3q+2-4t-2u} z^{r\,t+3u}$$

or

$$y^{2(q+1)-4r-2s} z^{q+r\,r+3s} = y^{2(q+1)-4t-2u} z^{q+r\,t+3u}.$$

Both equalities yield the following two equations

$$-4r - 2s = -4t - 2u \text{ and } nr + 3s = nt + 3u.$$

For $n \neq 6$, $(12 - 2n) \neq 0$ and $s = u$. Plugging back into either of the above two equations we see $r = t$. Thus for $\alpha \in k$ and $n \neq 6$,

$$y^q((y^2 + z^3)^2 + \alpha z^r)^{\frac{q+1}{2}} \text{ and } z^q((y^2 + z^3)^2 + \alpha z^r)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

For $n = 6$ we note that $x^2 - ((y^2+z^3)^2+\alpha z^6)$ is a quasihomogeneous polynomial with degrees given by $\deg(x) = 6$, $\deg(y) = 3$ and $\deg(z) = 2$ satisfying $\deg(xy) =$

$9 < 6+4 = \deg(y^2)+\deg(z^2)$ and $\deg(xz) = 8 < 6+4 = \deg(y^2)+\deg(z^2)$. Thus by Theorem 3.23 $\tau = \mathfrak{m}$ and

$$y^q((y^2+z^3)^2+\alpha z^6)^{\frac{q+1}{2}} \text{ and } z^q((y^2+z^3)^2+\alpha z^6)^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}).

Now suppose $\alpha \notin k$ and α contributes a monomial of the form

$$y^{2q-3}z^{\frac{3(q+1)}{2}+r} \text{ or } y^{2q-2}z^{q+r}$$

Since $\alpha = \Sigma_{k \geq 0}\alpha_k z^k$ then for some nonzero k and some pair (t, u)

$$y^{2q-3}z^{\frac{3(q+1)}{2}+r} = y^{3q+2-4t-2u}z^{(r+k)t+3u}$$

or

$$y^{2q-2}z^{q+r} = y^{2(q+1)-4t-2u}z^{q+(r+k)t+3u}.$$

Both equalities imply

$$-4-2s = -4t-2u \text{ and } n+3s = (n+k)t+3u,$$

for $s = 0$ or $\frac{q+1}{2}$. As long as $n > 6$,

$$t = \frac{2n-12}{2(n+k)-12)},$$

which is only an integer for $k = 0$ implies $t = 1$ and $u = s$. Thus

$$y^q((y^2+z^i)^2+\alpha z^r)^{\frac{q+1}{2}} \text{ and } z^q((y^2+z^i)^2+\alpha z^r)^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}).

For $n = 6$, as above

$$(2n - 12) = (2(n + k) - 12)t$$

which implies $0 = 2kt$. Suppose $k \neq 0$, then $t = 0$ and $u = \frac{q+5}{2}$. Note the binomial coefficient when $(t, u) = (0, \frac{q+5}{2})$ is congruent to 0 mod p since

$$\binom{q+1}{r} \equiv 0 \bmod p$$

for $r \notin 0, 1, q, q+1$. Hence the coefficient of the monomial

$$y^{2q-3}z^{\frac{3(q+1)}{2}+r} \text{ or } y^{2q-2}z^{q+r}$$

remains nonzero modulo p.

For $n = 5$, Lemma 3.36 implies that there exists an automorphism taking $(y^2 + z^i)^2 + \alpha z^5$ to the form $y^4 + yz^4 + \beta z^5$ for β a unit but this is case E.III. This gives case 11).

E.V) By Remark 3.24 we will show that

$$y^q((y^2 + z^i)^2 + \alpha yz^r)^{\frac{q+1}{2}} \text{ and } z^q((y^2 + z^i)^2 + \alpha yz^r)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}),$$

for $i = 3$. To do this we look at the monomials in the expansion of

$$y^q((y^2 + z^i)^2 + \alpha yz^r)^{\frac{q+1}{2}} \text{ and } z^q((y^2 + z^i)^2 + \alpha yz^r)^{\frac{q+1}{2}}.$$

As long as $\alpha \in k$ the monomials in the expansion of

$$y^q((y^2 + z^i)^2 + \alpha yz^r)^{\frac{q+1}{2}} \text{ and } z^q((y^2 + z^i)^2 + \alpha yz^r)^{\frac{q+1}{2}}$$

have the form

$$y^{3q+2-3r-2s}z^{rr+is} \text{ and } y^{2(q+1)-3r-2s}z^{q+rr+is}$$

for $0 \leq r \leq \frac{q+1}{2}$ and $0 \leq s \leq q+1-2r$. Setting $r=1$ and $s=\frac{q+1}{2}$ in the first expression above, the monomial we obtain is

$$y^{2q-2}z^{\frac{i(q+1)}{2}+r} \notin (y^{2q}, z^{2q})$$

for $i=3$ and as long as $q > 2n+3$ but it is always contained in (y^{2q}, z^{2q}) for $i \geq 4$. Note by Lemma 3.29 the coefficient of $\binom{q-1}{\frac{q+1}{2}} \not\equiv 0 \bmod p$.

Setting $r=1$ and $s=0$ in the second expression above the monomial we obtain is

$$y^{2q-1}z^{q+r}$$

which is not contained in (y^{2q}, z^{2q}) for $q > n$.

Note that the monomials are uniquely determined by r and s. Suppose for some t and u

$$y^{2(q+1)-3r-2s}z^{rr+3s} = y^{2(q+1)-3t-2u}z^{rt+3u}.$$

This equality yields the following two equations

$$-3r-2s = -3t-2u \text{ and } nr+3s = nt+3u.$$

For $n > 3$, $(9-2n) \neq 0$. Thus $s=u$ which in turn yields $r=t$.

Now suppose $\alpha \notin k$ and α contributes a monomial of the form

$$y^{2q-2}z^{\frac{3(q+1)}{2}+r} \text{ or } y^{2q-1}z^{q+r}.$$

Since $\alpha = \Sigma_{k\geq 0}\alpha_k z^k$ then for some nonzero k and some pair (t, u),

$$y^{2q-2}z^{\frac{3(q+1)}{2}+r} = y^{3q+2-3t+2u}z^{(r+k)t+3u} \text{ or } y^{2q-1}z^{q+r} = y^{2(q+1)-3t+2u}z^{q+(r+k)t+3u}.$$

Both of the above equalities imply

$$-3 - 2s = -3t - 2u \text{ and } n + 3s = (n + k)t + 3u,$$

where $s = 0$ or $\frac{q+1}{2}$. Thus $t = 1$ and $u = s$ implying that

$$y^q((y^2 + z^i)^2 + \alpha yz^r)^{\frac{q+1}{2}} \text{ and } z^q((y^2 + z^i)^2 + \alpha yz^r)^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). This gives case 10).

E.VI) We first reduce to the case that $i = 3$ by noting that some of the monomials of the expansion of

$$((y^2 + z^i) + \alpha yz^r + \beta z^m)^{\frac{q+1}{2}}$$

are just monomials of the expansion of

$$((y^2 + z^i) + \alpha yz^r)^{\frac{q+1}{2}}.$$

Thus $i = 3$. By Lemma 3.37 this case reduces to E.IV). $\square$

3.6 Classification – the case when $a \in (y,z)^5 \backslash (y,z)^6$

When the a a quintic plus higher order terms there will be at least as many isomorphism classes as there are distinct linear factorizations of quintics. These factorizations are among the following:

A) $yz(y+z)(y+\lambda z)(y+\mu z)$, μ, λ units,

B) $y^2z(y+z)(y+\lambda z)$, λ a unit,

C) $y^2z^2(y+z)$,

D) $y^3z(y+z)$,

E) y^3z^2,

F) y^4z,

G) y^5.

Using the lemmas in the appendix if we have a quintic in one of the above forms plus higher order terms we can make a change of variables absorbing units into x to rewrite these quintics as:

A) $yz(y+z)(y+\lambda z)(y+\mu z)$, μ, λ units,

B) $(y^2+\alpha z^r)z(y+z)(y+\lambda z)$, α, λ units, $n \geq 3$,

C) $(y^2+\alpha z^r)(z^2+\beta y^m)(y+z)$, α, β units, $n, m \geq 3$,

D) $(y^3+cz^i)z(y+z)$, $c \in (y,z)$, $i \geq 3$,

E) $(y^3+cz^i)z^2$, $c \in (y,z)$, $i \geq 3$,

F) $((y^2+cz^i)^2+dz^r)z$, c a unit, $d \in (y,z)$ $n > i \geq 3$,

G) $y^5+\alpha y^3z^i+\beta y^2z^j+\gamma yz^k+\delta z^l$.

We can easily check that if R is defined by $k[[x,y,z]]/(x^2-a)$ where a is in the form A, B or C then $y^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ and $z^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$. In cases D and E, we can use many of the lemmas in chapter 8 to analyze which $c \in (y,z)$ will force $y^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$ and $z^q a^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$. The analysis of a in the form given by case F is similar to E in the previous chapter.

I) $(y^4 + \alpha z^i)z$, where α is a unit in $k[[y,z]]$,

II) $(y^4 + \alpha y z^i)z$, where α is a unit in $k[[y,z]]$,

III) $(y^4 + \alpha y z^i + \beta z^j)z$, where α and β are units in $k[[y,z]]$,

IV) $((y^2 + z^i)^2 + \alpha z^r)z$, where α is a unit in $k[[z]]$,

V) $((y^2 + z^i)^2 + \alpha y z^r)z$, where α is a unit in $k[[z]]$,

VI) $((y^2 + z^i)^2 + \alpha y z^r + \beta z^m)z$, where α and β are units in $k[[z]]$.

However IV-VI are isomorphic to powerseries that were already determined by C. Case G breaks down into the following:

I) $y^5 + \alpha z^i$,

II) $y^5 + \alpha y z^i$,

III) $y^5 + \alpha y z^i + \beta z^j$,

IV) $y^5 + \alpha y^2 z^i + \beta z^j$,

V) $y^5 + \alpha y^3 z^i + \beta z^j$,

VI) $y^5 + \alpha y^2 z^i + \beta y z^j$,

VII) $y^5 + \alpha y^3 z^i + \beta y z^j$,

VIII) $y^5 + \alpha y^2 z^i + \beta y z^j + \gamma z^k$,

IX) $y^5 + \alpha y^3 z^i + \beta y z^j + \gamma z^k$,

X) $y^5 + \alpha y^3 z^i + \beta y^2 z^j + \gamma z^k$,

XI) $y^5 + \alpha y^3 z^i + \beta y^2 z^j + \gamma y z^k + \delta z^l$.

We will use the following lemmas to analyze some of the above cases.

Lemma 3.39 *Suppose the characteristic of k is greater than i and let $R = k[[x, y, z]]/(x^2 - a)$ or $R = k[[x, y, z]]/(x^2 - a')$ where $a = y^5 + \alpha z^i$ and $a' = y^5 + \alpha y z^i$ and α is a unit in $k[[y, z]]$ in both cases. Then there exists an automorphism of $k[[x, y, z]]$ taking $x^2 - (y^5 + z^i)$ to $x^2 - a$ and $x^2 - (y^5 + yz^i)$ to $x^2 - a'$.*

Proof. Since the characteristic of R is greater than i and if α_k is the component of $\alpha \in k$ then $t^i - \alpha_k$ has a root in $k[t]$. Since $k[[x, y, z]]$ is complete then Hensel's lemma implies that $t^i - \alpha$ has a root in $k[[x, y, z]][t]$. Suppose that θ is a root of

$t^i - \alpha$. Setting $z_1 = \theta z$, we can rewrite

$$y^5 + \alpha z^i = y^3 + z_1^i$$

and

$$y^5 + \alpha y z^i = y^3 + y z_1^i.$$

Thus the automorphism $s : k[[x, y, z]] \to k[[x, y, z]]$ given by $s(x) = x, s(y) = y$ and $s(z) = \theta z$ maps $(x^2 - (y^5 + z^i))$ to $(x^2 - a)$ and $(x^2 - (y^5 + yz^i))$ to $(x^2 - a')$. □

Lemma 3.40 *Suppose the characteristic of k is equal to $p > i$ and let $R = k[[y, z]]$. Suppose $a = y^5 + yz^i + \gamma z^{i+j}$ where $j \geq i$ and $f = y^5 + yz^i$. Then there exists an automorphism of R sending f to a.*

Proof. Set $y_1 = y + \gamma z^j$ and suppose α and β be the transformed coefficients of yz^i and z^{5j}. Then

$$y^5 + yz^i + \gamma z^{i+j} = y_1^5 + \alpha y_1 z^i + \beta z^{5j}.$$

Let $s_1 : k[[y, z]] \to k[[y, z]]$ be the automorphism defined by $s_1(y) = y + \gamma z^j$ and $s_2(z) = z$.

Set $z_1 = \alpha^{\frac{1}{i}} z$ and suppose that β_1 is the transformed coefficients of z^{5j}. Then

$$y_1^5 + \alpha y_1 z^i + \beta z^{5j} = y_1^5 + y_1 z^i + \beta_1 z^{5j}.$$

Let $s_2 : k[[y, z]] \to k[[y, z]]$ be the automorphism defined by $s_2(y) = y$ and $s_2(z) = \alpha^{\frac{1}{i}} z$.

But $5j > i+j$ and $z^{5j} \in \mathfrak{m}(\Delta(f))^2$. Thus by [3, Theorem B], there exists an automorphism of R sending f to a. □

Lemma 3.41 *Suppose the characteristic of k is equal to $p > i$ and let $R = k[[y,z]]$. Suppose $a = y^5 + \alpha y^3 z^3 + \beta y^2 z^4 + yz^5 + \gamma z^6$ where either α, β and γ are units or one of α or β is 0 and $f = y^5 + \delta y^3 z^3 + \epsilon y^2 z^4 + z^6$ where δ and ϵ are units depending on α, β and γ. Then there exists an automorphism taking f to a.*

Proof. Set $z_1 = \gamma^{\frac{1}{6}} z$, $\alpha_1 = \frac{\alpha}{\gamma^{\frac{3}{6}}}$, $\beta_1 = \frac{\beta}{\gamma^{\frac{4}{6}}}$ and $\gamma_1 = \frac{1}{\gamma^{\frac{5}{6}}}$. Then

$$y^5 + \alpha y^3 z^3 + \beta y^2 z^4 + yz^5 + \gamma z^6 = y^5 + \alpha_1 y^3 z_1^3 + \beta_1 y^2 z_1^4 + \gamma_1 y z_1^5 + z_1^6.$$

Define $s_1 : k[[y,z]] \to k[[y,z]]$ by $s_1(y) = y$ and $s_1(z) = \gamma^{\frac{1}{6}} z$.

Set $z_2 = z_1 + \frac{\gamma_1}{6} y$. Let α_2, β_2, γ_2 and δ_2 be the transformed unit coefficients of $y^3 z_2^3$, $y^2 z_2^4$, $y^4 z_2^2$ and y^5. Then

$$y^5 + \alpha_1 y^3 z_1^3 + \beta_1 y^2 z_1^4 + \gamma_1 y z_1^5 + z_1^6 = \delta_2 y^5 + \gamma_2 y^4 z_2^2 + \alpha_2 y^3 z_2^3 + \beta_2 y^2 z_2^4 + z_2^6.$$

Define $s_2 : k[[y,z]] \to k[[y,z]]$ by $s_2(y) = y$ and $s_2(z) = z + \frac{\gamma_1}{6} y$.

Set $y_1 = \delta_2^{\frac{1}{5}} y$. Let α_3, β_3 and γ_3 be the transformed unit coefficients of $y_1^3 z_2^3$, $y_1^2 z_2^4$ and $y_1^4 z_2^2$. Then

$$\delta_2 y^5 + \gamma_2 y^4 z_2^2 + \alpha_2 y^3 z_2^3 + \beta_2 y^2 z_2^4 + z_2^6 = y_1^5 + \gamma_3 y_1^4 z_2^2 + \alpha_3 y_1^3 z_2^3 + \beta_3 y_1^2 z_2^4 + z_2^6.$$

Define $s_3 : k[[y,z]] \to k[[y,z]]$ by $s_3(y) = \delta_2^{\frac{1}{5}} y$ and $s_3(z) = z$.

Set $y_2 = y_1 + \frac{\gamma_3}{5} z_2^2$. Let α_4, β_4 and γ_4 be the transformed unit coefficients of $y_2^3 z_2^3$, $y_2^2 z_2^4$ and z_2^6. Then

$$y_1^5 + \gamma_3 y_1^4 z_2^2 + \alpha_3 y_1^3 z_2^3 + \beta_3 y_1^2 z_2^4 + z_2^6 = y_2^5 + \alpha_4 y_2^3 z_2^3 + \beta_4 y_2^2 z_2^4 + \gamma_4 z_2^6.$$

Define $s_4 : k[[y,z]] \to k[[y,z]]$ by $s_4(y) = y + \frac{\gamma_3}{5}z^2$ and $s_4(z) = z$.

Set $z_3 = \gamma_4^{\frac{1}{6}} z_2$. Let α_5 and β_5 be the transformed unit coefficients of $y_2^3 z_3^3$ and $y_2^2 z_3^4$. Then

$$y_2^5 + \alpha_4 y_2^3 z_2^3 + \beta_4 y_2^2 z_2^4 + \gamma_4 z_2^6 = y_2^5 + \alpha_5 y_2^3 z_3^3 + \beta_5 y_2^2 z_3^4 + z_3^6.$$

Define $s_5 : k[[y,z]] \to k[[y,z]]$ by $s_5(y) = y$ and $s_5(z) = \gamma_4^{\frac{1}{6}} z$.

Then $s : k[[y,z]] \to k[[y,z]]$ defined by $s = s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_5$ defines an automorphism taking f to a. □

We are now ready to classify those $a \in (y,z)^5 \backslash (y,z)^6$.

Theorem 3.42 *Suppose the characteristic of k is $p > 6$ and let $R = k[[x,y,z]]/(x^2 - a)$ where $a \in (y,z)^5 \backslash (y,z)^6$, then $\mathfrak{m}$ is the test ideal if and only if after a change of variables we can write a in one of the following forms:*

1) $a = yz(y+z)(y+\lambda z)(y+\mu z)$, λ *a unit,*

2) $a = (y^2+\alpha z^r)z(y+z)(y+\lambda z)$, α *a unit and* $n \geq 3$,

3) $a = (y^2+\alpha z^r)(z^2+\beta y^m)(y+z)$, α, β *units and* $n, m \geq 3$,

4) $a = (y^3+z^i)z(y+z)$, $4 \leq i < 6$,

5) $a = (y^3+yz^3)z(y+z)$,

6) $a = (y^3+yz^i+\alpha z^{i+j})z(y+z)$, $3 \leq i < 5$, $1 \leq j < i$, α *a unit,*

7) $a = (y^3+z^i)(z^2+\alpha y^m)$, $4 \leq i < 6$ *and* $m \geq 3$, α *a unit,*

8) $a = (y^3+yz^i)(z^2+\alpha y^m)$, $i = 3$ *and* $m \geq 3$, α *a unit,*

9) $a = (y^3+yz^i+\beta z^{i+j})(z^2+\alpha y^m)$, $3 \leq i < 5$, $1 \leq j < i$, α, β *units,*

10) $a = (y^4+z^5)z$,

11) $a = (y^4+yz^4)z$,

12) $a = (y^4+yz^i+\alpha z^{i+j})z$, $i = 4$ *and* $1 \leq j < i$, α *a unit,*

13) $a = y^5+z^6$,

14) $a = y^5+yz^5$,

15) $a = y^5+yz^5+\alpha z^{5+j}$, $1 \leq j < i$, α *a unit,*

16) $a = y^5+\alpha y^2z^j+z^6$, $4 \leq j \leq 5$, α *a unit,*

17) $a = y^5+\alpha y^3z^j+z^6$, $3 \leq j \leq 5$, α *a unit,*

18) $a = y^5+\alpha y^2z+yz^5$, α *a unit,*

19) $a = y^5+\alpha y^3z^j+yz^5$, $3 \leq j \leq 4$, α *a unit,*

20) $a = y^5+\alpha y^3z^j+\beta y^2z^k+z^6$, $3 \leq j \leq 4$, $4 \leq k \leq 5$, *and* $j < k$, α, β *units,*

21) $a = y^5+\alpha y^3z^k+\beta yz^5+z^{5+j}$, $3 \leq k \leq 4$, $2 \leq j < i$, α, β *units,*

22) $a = y^5+\alpha y^2z^4+\beta yz^5+z^{5+j}$, $2 \leq j < i$, α, β *units,*

23) $a = y^5+\alpha y^3z^3+\beta y^2z^4+\gamma yz^5+z^{5+j}$, $2 \leq j < i$ α, β, γ *units,.*

Proof. If $a \in (y,z)^5 \backslash (y,z)^6$, then the initial form of a will either be:

A) a product of five independent linear terms, $yz(y+z)(y+\lambda z)(y+\mu z)$,

B) a product of a square and three independent linear terms, $y^2z(y+z)(y+\lambda z)$,

C) a product of two independent squares and a linear term, $y^2z^2(y+z)$,

D) a product of a cube and two independent linear terms, $y^3z(y+z)$,

E) a product of a cube and a square, y^3z^2,

F) a product of a fourth power and a linear term, y^4z,

G) a fifth power of a linear term, y^5.

Each initial form will give us at least one and in some cases, more than one isomorphism class of rings that have test ideal equal to the maximal ideal. To check each isomorphism class has the test ideal equal to the maximal ideal, by theorem 3.24 we need only check that $a^{\frac{q+1}{2}} \in (y^q, z^q)$ and $a^{\frac{q+1}{2}}y^q \notin (y^{2q}, z^{2q})$ and $a^{\frac{q+1}{2}}z^q \notin (y^{2q}, z^{2q})$. Theorem 3.25 guarantees that $a^{\frac{q+1}{2}} \in (y^q, z^q)$ since $a \in (y,z)^5 \cap k[[y,z]]$.

A) If the initial form is $yz(y+z)(y+\lambda z)(y+\mu z)$, then by Lemma A.11 $yz(y+z)(y+\lambda z)(y+\mu z)+b$ where $b \in (y,z)^6$ can be rewritten after some change of variables as $yz(y+z)(y+\lambda z)(y+\mu z)$ where λ and μ are units in $k[[y,z]]$.

Assume λ and μ are in k for the moment. Note that $x^2-a = x^2-yz(y+z)(y+\lambda z)(y+\mu z)$ is a quasihomogeneous polynomial with $\deg(x) = 5$ and $\deg(y) = \deg(z) = 2$. By Corollary 3.23 $\tau = \mathfrak{m}$ if and only if $\deg(x) = 5 \geq 2+2 = \deg(y)+\deg(z)$, $\deg(xy) = 7 < 4+4 = \deg(y^2)+\deg(z^2)$ and $\deg(xz) = 7 < 4+4 = \deg(y^2)+\deg(z^2)$.

If λ and μ were not contained in k,

$$\lambda = \lambda_0 + \Sigma_{i,j\geq 1}\lambda_{ij}y^iz^j \text{ and } \mu = \mu_0 + \Sigma_{i,j\geq 1}\mu_{ij}y^iz^j.$$

Noting that the monomials in the expansion of

$$y^q(yz(y+z)(y+\lambda_{00}z)(y+\mu_{00}z))^{\frac{q+1}{2}} \text{ and } z^q(yz(y+z)(y+\lambda_{00}z)(y+\mu_{00}z))^{\frac{q+1}{2}}$$

all have degree $\frac{7q+5}{2}$, then if we multiply any monomial in the expansion of either of the two expressions above by y^iz^j this will only give us a monomial with a larger degree. Thus no terms in the expression of λ or μ will cancel out any monomial in the expansion of either of the above expressions that is not contained in (y^{2q}, z^{2q}). Hence by Remark 3.24 we have case 1).

B) If the initial form is $y^2z(y+z)(y+\lambda z)$, then by Lemma A.12, $y^2z(y+z)+b$ where $b \in (y,z)^6$ can be rewritten after some change of variables as $(y^2+\alpha z^r)z(y+z)(y+\lambda z)$ where α and λ are units in $k[[y,z]]$ and $n \geq 3$. Assume $\alpha, \lambda \in k$ for the moment. By Remark 3.24 we need to show that

$$y^q((y^2+\alpha z^r)z(y+z)(y+\lambda z))^{\frac{q+1}{2}} \text{ and } z^q((y^2+\alpha z^r)z(y+z)(y+\lambda z))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

To do this we exhibit that there exists a monomial in the expansion of both

$$y^q((y^2+\alpha z^r)z(y+z)(y+\lambda z))^{\frac{q+1}{2}} \text{ and } z^q((y^2+\alpha z^r)z(y+z)(y+\lambda z))^{\frac{q+1}{2}}$$

that is not contained in (y^{2q}, z^{2q}). The monomials are of the form

$$y^{3q+2)-2r-s-t}z^{\frac{q+1}{2}+rr+s+t} \text{ and } y^{2(q+1))-2r-s-t}z^{\frac{3q+1}{2}+rr+s+t}.$$

Taking $r = 1, s = \frac{q+1}{2}$ and $t = \frac{q+1}{2}$ in the first expression above and $r = 0, t = 0$ and $s = \frac{q-3}{2}$ in the second expression above gives us the monomials

$$y^{2q-1} z^{\frac{3(q+1)}{2}+r} \text{ and } y^{\frac{3q+7}{2}} z^{2q-1}.$$

which are not in (y^{2q}, z^{2q}) for $q > 2n + 3$.

Notice if α and λ were not contained in k then write

$$\alpha = \Sigma_{i,j\geq 0} \alpha_{ij} y^i z^j \text{ and } \lambda = \Sigma_{i,j\geq 0} \lambda_{kl} y^k z^l.$$

Suppose that α and λ contribute another monomial in the form

$$y^{2q-1} z^{\frac{3(q+1)}{2}+r}.$$

Then for some r, s and t we have the following equalities:

$$y^{2q-1} z^{\frac{3(q+1)}{2}+r} = y^{3q+2+(i-2)r-s+(k-1)t} z^{\frac{q+1}{2}+(r+j)r+s+(l+1)t}$$

or

$$y^{\frac{3q+7}{2}} z^{2q-1} = y^{2(q+1)+(i-2)r-s+(k-1)t} z^{\frac{3q+1}{2}+(r+j)r+s+(l+1)t}.$$

The first equality implies

$$(i-2)r - s + (k-1)t = q - 1 \text{ and } (n+j)r + s + (l+1)t = n + q + 1$$

and the second implies

$$(i-2)r - s + (k-1)t = -\frac{q-3}{2} \text{ and } (n+j)r + s + (l+1)t = \frac{q-3}{2}.$$

In each system of equations above, adding the two equations together we see that

$$n - 2 = (n-2+i+j)r + (k+l)t \text{ and } 0 = (n-2+i+j)r + (k+l)t.$$

We note that for i, j, k, l, t and $r > 0$ then

$$n-2 < (n-2+i+j)r + (k+l)t.$$

The only possible solution would be when $r = 0$ and $(k+l)t = n-2$ or 0. Substituting $r = 0$ and $t = \frac{r-2}{k+l}$ into the first system of equations yields $(k+l)s = n(k-1)+2(l+1)+(k+l)(q+1)$ which is a contradiction since $0 \le s \le \frac{q+1}{2}$. Thus $r = 1$, $s = t = \frac{q+1}{2}$ and $i, j, k, l = 0$. Substituting $r = 0$ and $t = 0$ into the second system yields $s = \frac{q-3}{2}$. Thus no terms in the expansion of α or λ will contribute any more monomials of the form

$$y^{2q-1}z^{\frac{3(q+1)}{2}+r} \text{ or } y^{\frac{3q+7}{2}}z^{2q-1}.$$

Thus

$$y^q((y^2+\alpha z^r)z(y+z))^{\frac{q+1}{2}} \text{ and } z^q((y^2+\alpha z^r)z(y+z))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence by Remark 3.24 we have case 2).

C) If the initial form is $y^2z^2(y+z)$, then by Lemma A.13, $y^2z^2(y+z)+b$ where $b \in (y,z)^6$ can be rewritten after some change of variables as $(y^2+\alpha z^r)(z^2+\beta y^m)(y+z)$ where α and β are units in $k[[y,z]]$ and $n, m \ge 3$. Assume $\alpha, \beta \in k$ for the moment. By Remark 3.24 we show that

$$y^q((y^2+\alpha z^r)(z^2+\beta y^m)(y+z))^{\frac{q+1}{2}} \text{ and } z^q((y^2+\alpha z^r)(z^2+\beta y^m)(y+z))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

by exhibiting that there exists a monomial in the expansion of both

$$y^q((y^2+\alpha z^r)(z^2+\beta y^m)(y+z))^{\frac{q+1}{2}} \text{ and } z^q((y^2+\alpha z^r)(z^2+\beta y^m)(y+z))^{\frac{q+1}{2}}$$

that is not contained in (y^{2q}, z^{2q}). The monomials are of the form

$$y^{\frac{5q+3}{2}-2r+ms-t}z^{q+1-2s+rr+t} \text{ and } y^{\frac{3(q+1)}{2}-2r+ms-t}z^{2q+1-2s+rr+t}.$$

Taking $r = 1, s = 0$ and $t = \frac{q+1}{2}$ in the first expression and $r = 0, s = 1$ and $t = 0$ gives us the monomials

$$y^{2q-1}z^{\frac{3(q+1)}{2}+r} \text{ and } y^{\frac{3(q+1)}{2}+m}z^{2q-1}$$

which are not in (y^{2q}, z^{2q}) for $q > \max(n, m)$.

Notice if α or β were not contained in k then write

$$\alpha = \Sigma_{i,j \geq 0}\alpha_{ij}y^i z^j \text{ and } \beta = \Sigma_{i,j \geq 0}\beta_{kl}y^k z^l.$$

Suppose that α and β contribute another monomial in the form

$$y^{2q-1}z^{\frac{3(q+1)}{2}+r} \text{ or } y^{\frac{3(q+1)}{2}+m}z^{2q-1}.$$

Then for some r, s and t we have the two following system of equations

$$(i-2)r + (m+k)s - t = -2 - \frac{q+1}{2} \text{ and } (n+j)r + (l-2)s + t = \frac{q+1}{2} + n$$

and

$$(i-2)r + (m+k)s - t = m \text{ and } (n+j)r + (l-2)s + t = -2.$$

Note that both

$$n-2 \text{ and } m-2 < (n-2+i+j)r + (m+k+l-2)s$$

for $i, j, k, l, r, s > 0$. Hence, the only possible solution besides $r = 1$ and $i, j, k, l, s = 0$ is $(m+k+l-2)s = n-2$ or $(n+i+j-2)r = m-2$. Substituting $r = 0$

and $s = \frac{r-2}{m+k+l-2}$ into the first system of equations yields $(m+k+l-2)t = (m+k)n + 2(l-2) + \frac{(m+k+l-2)(q+1)}{2}$ which is a contradiction since $0 \leq t \leq \frac{q+1}{2}$. Thus $r = 1$, $t = \frac{q+1}{2}$ and $i, j, k, l = 0$. Substituting $s = 0$ and $r = \frac{m-2}{r+i+j-2}$ into the first system of equations yields $(n+i+j-2)t = (n+j)m + 2(k-2) + \frac{(r+i+j-2)(q+1)}{2}$ which is a contradiction since $0 \leq t \leq \frac{q+1}{2}$. Thus $s = 1$, $t = 0$ and $i, j, k, l = 0$. Thus no terms in the expansion of α or β will contribute any more monomials of the form

$$y^{2q-1} z^{\frac{3(q+1)}{2}+r} \text{ or } y^{\frac{3(q+1)}{2}+m} z^{2q-1}.$$

Thus

$$y^q((y^2+\alpha z^r)(z^2+\beta y^m)(y+z))^{\frac{q+1}{2}} \text{ and } z^q((y^2+\alpha z^r)(z^2+\beta y^m)(y+z))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence by Remark 3.24, we have case 3).

D) If the initial form is $y^3 z(y+z)$, then by Lemma A.14, $y^3 z(y+z) + b$ where $b \in (y, z)^6$ can be rewritten after some change of variables as $(y^3 + az^3)z(y+z)$ where $a \in (y, z) \cap k[[y, z]]$. By the Weierstrass Preparation Theorem, $y^3 + az^3$ can be replaced by $u(y, z)(y^3 + a(z)y + b(z))$ where $u(y, z)$ is a unit and $a(z), b(z) \in k[[z]]$. To determine the distinct isomorphism classes which stem from this form we break this situation down into the three cases: I) $a(z) = 0$, II) $b(z) = 0$ and III) $a(z) \neq 0$ and $b(z) \neq 0$. Or in otherwords,

I) $z(y+z)(y^3 + \alpha z^i)$, where α is a unit in $k[[z]]$,

II) $z(y+z)(y^3 + \alpha y z^i)$, where α is a unit in $k[[z]]$,

III) $z(y+z)(y^3 + \alpha y z^i + \beta z^j)$, where α and β are units in $k[[z]]$.

D.I) We need only check that

$$y^q(z(y+z)(y^3+\alpha z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y+z)(y^3+\alpha z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

as long as $4 \leq i < 6$.

First let us assume that $\alpha \in k$. The monomials in the expansion of

$$y^q(z(y+z)(y^3+\alpha z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y+z)(y^3+\alpha z^i))^{\frac{q+1}{2}}$$

are in the form

$$y^{3q+2-3r-s}z^{\frac{q+1}{2}+ir+s} \text{ and } y^{2(q+1)-3r-s}z^{\frac{3q+1}{2}+ir+s},$$

for $0 \leq r, s \leq \frac{q+1}{2}$. Suppose $s = \frac{q+1}{2}$ and $\frac{5q+3}{2} - 3r < 2q$ in the first expression. By Lemma 3.28, there exists an r such that

$$\binom{\frac{q+1}{2}}{r} \neq 0.$$

Then $r > \frac{q+3}{6}$, which implies $q+1+ir > \frac{(i+6)q+(3i+6)}{6}$. Suppose $\frac{(i+6)q+(3i+6)}{6} > 2q$ for all q. Then $(i+6)q+3i+6 > 12q$ or $3i+3 > (6-i)q$. For $i < 6$ we can choose $q > \frac{3i+3}{9-i}$ which implies that with such r and $s = \frac{q+1}{2}$ then

$$y^{\frac{5q+3}{2}-3r}z^{q+1+ir} \notin (y^{2q}, z^{2q}).$$

Taking $r = 0$ and $s = \frac{q-3}{2}$ in the second, the monomial we obtain is

$$y^{\frac{3q+7}{2}}z^{2q-1}$$

which is clearly not in (y^{2q}, z^{2q}) for $q > 7$.

Suppose $\alpha \notin k$, then $\alpha = \Sigma_{k,l \geq 0} \alpha_{kl} y^k z^l$. Suppose for $k, l \neq 0$ and some pair (t, u) there exists some monomial equal to

$$y^{\frac{5q+3}{2} - 3r} z^{q+1+ir}.$$

In other words,

$$y^{\frac{5q+3}{2} - 3r} z^{q+1+ir} = y^{3q+2+(k-3)t-u} z^{\frac{q+1}{2}+(i+l)t+u}.$$

This equality yields the following two equations:

$$-\frac{q+1}{2} - 3r = (k-3)t - u \text{ and } \frac{q+1}{2} + ir = (i+l)t + u.$$

Hence, $i \geq 4$ and $u - \frac{q+1}{2} \leq 0$ so $u = \frac{q+1}{2}$ and $t = r$. Thus α contributes no other monomial of the form

$$y^{\frac{5q+3}{2} - 3r} z^{q+1+ir};$$

hence,

$$y^q (z(y+z)(y^3 + \alpha z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Similarly if there exists another monomial equal to

$$y^{\frac{3q+7}{2}} z^{2q-1}$$

then there exists some (t, u) such that

$$y^{\frac{3q+7}{2}} z^{2q-1} = y^{2(q+1)+(k-3)t-u} z^{\frac{3q+1}{2}+(i+l)t+u}.$$

This equality yields the following two equations

$$-\frac{q-3}{2} = (k-3)t - u \text{ and } \frac{q-3}{2} = (i+l)t + u.$$

Adding the two together yields $0 = (k-3+i+l)t$ and $k-3+i+l > 0$ since $i \geq 4$ implies that $t = 0$. Hence, α contributes no such monomial of the form

$$y^{\frac{3q+7}{2}} z^{2q-1} \text{ and } z^q(z(y+z)(y^3+\alpha z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence by Remark 3.24 we have case 4).

D.II) We need only check that

$$y^q(z(y+z)(y^3+\alpha yz^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y+z)(y^3+\alpha yz^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

as long as $3 \leq i < 4$.

First let us assume that $\alpha \in k$. The monomials in the expansion of

$$y^q(z(y+z)(y^3+\alpha yz^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y+z)(y^3+yz^i))^{\frac{q+1}{2}}$$

are in the form

$$y^{3q+2-2r-s} z^{\frac{q+1}{2}+ir+s} \text{ and } y^{2(q+1)-2r-s} z^{\frac{3q+1}{2}+ir+s},$$

for $0 \leq r, s \leq \frac{q+1}{2}$. Suppose $s = \frac{q+1}{2}$ and

$$\frac{5q+3}{2} - 2r < 2q$$

in the first expression. By Lemma 3.27 there exists an r such that $\binom{\frac{q+1}{2}}{r} \neq 0$. Then $r > \frac{q+3}{4}$, which implies $q+1+ir > \frac{(i+4)q+(3i+4)}{4}$. Suppose $\frac{(i+4)q+(3i+4)}{4} > 2q$ for all q. Then $(i+4)q+3i+4 > 8q$ or

$$3i+4 > (4-i)q.$$

For $i < 4$ we can choose $q > \frac{3i+4}{4-i}$. Thus

$$y^q(z(y+z)(y^3+\alpha y z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Taking $r = 0$ and $s = \frac{q-3}{2}$ in the second expression, the monomial we obtain is

$$y^{\frac{3q+7}{2}} z^{2q-1}$$

which is clearly not in (y^{2q}, z^{2q}) for $q > 4$.

Suppose $\alpha \notin k$, then $\alpha = \Sigma_{k,l \geq 0} \alpha_{kl} y^k z^l$. Suppose for $k, l \neq 0$ and some pair (t, u) there exists some monomial equal to

$$y^{\frac{5q+3}{2}-2r} z^{q+1+ir}.$$

In other words,

$$y^{\frac{5q+3}{2}-2r} z^{q+1+ir} = y^{3q+2+(k-2)t-u} z^{\frac{q+1}{2}+(i+l)t+u}.$$

This equality yields the following two equations:

$$-\frac{q+1}{2} - 2r = (k-2)t - u \text{ and } \frac{q+1}{2} + ir = (i+l)t + u.$$

Hence, $i \geq 3$ and $u - \frac{q+1}{2} \leq 0$ so $u = \frac{q+1}{2}$ and $t = r$ which implies α does not contribute another monomial of the form

$$y^{\frac{5q+3}{2}-2r} z^{q+1+ir}.$$

Hence,

$$y^q(z(y+z)(y^3+\alpha y z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Similarly if there exists another monomial of the form

$$y^{\frac{3q+7}{2}}z^{2q-1}$$

then there exists some (t, u) such that

$$y^{\frac{3q+7}{2}}z^{2q-1} = y^{2(q+1)+(k-2)t-u}z^{\frac{3q+1}{2}+(i+l)t+u}.$$

This equality yields the following two equations:

$$-\frac{q-3}{2} = (k-2)t - u \text{ and } \frac{q-3}{2} = (i+l)t + u.$$

Adding the two together yields $0 = (k-2+i+l)t$ and $k-2+i+l > 0$ since $i \geq 3$ implies that $t = 0$. Hence, α contributes no such monomial of the form

$$y^{\frac{3q+7}{2}}z^{2q-1}$$

and

$$z^q(z(y+z)(y^3+\alpha yz^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence by Remark 3.24 we have case 5).

D.III) We need only consider the case when $i < j$ since Lemma 3.31 implies that if $i \geq j$ then

$$k[[x,y,z]]/(x^2 - z(y+z)(y^3+\alpha yz^i+\beta z^j)) \cong k[[x,y,z]]/(x^2 - z(y+z)(y^3+z^j))$$

which was case 4) for $4 \leq j < 6$. For simplicity replace j by $i+j$ where $j \geq 1$.

By Remark 3.24 we show that the test ideal must be the maximal ideal by looking at the monomials in the expansion of

$$y^q(z(y+z)(y^3+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}}$$

and

$$z^q(z(y+z)(y^3+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}}.$$

Taking into account that

$$\alpha = \Sigma_{k,l\geq 0}\alpha_{kl}y^k z^l \text{ and } \beta = \Sigma_{m,r\geq 0}\alpha_{mr}y^m z^r$$

the monomials in the expansion of

$$y^q(z(y+z)(y^3+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}} \text{ and } z^q(z(y+z)(y^3+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}}$$

are in the form

$$y^{3q+2+(m-3)r+(k-2)s-t}z^{\frac{q+1}{2}+(i+j+r)r+(i+l)s+t}$$

and

$$y^{2(q+1)+(m-3)r+(k-2)s-t}z^{\frac{3q+1}{2}+(i+j+r)r+(i+l)s+t}.$$

Setting $s=0$ gives us monomials in the expansion of

$$y^q(z(y+z)(y^3+\beta z^{i+j}))^{\frac{q+1}{2}}$$

and we know from above that

$$y^q(z(y+z)(y^3+\beta z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $4 \leq i+j \leq 5$. Set $t = \frac{q+1}{2}$ and $m=0$ and suppose r is such that

$$y^{\frac{5q+3}{2}-3r}z^{q+1+(i+j)r}$$

is not contained in (y^{2q}, z^{2q}). Suppose there is another monomial of this form. Then there exists a triplet (u,v,w) such that

$$\frac{5q+3}{2} - 3r = 3q+2+(m-3)u+(k-2)v-w$$

and

$$\frac{q+1}{2} + (i+j)r = (i+j+n)u + (i+l)v + w.$$

For $3 \le i \le 4$ and $j = 1$, $(1+k)i + (k-2)j + 3l > 0$ and $3 - (i+j) < 0$ which implies $v = 0$ and $w = \frac{q+1}{2}$. Plugging $v = 0$ and $w = \frac{q+1}{2}$ into either of the above equations yields $u = r$ and $m = n = 0$. Thus in the case $3 \le i \le 4$ and $j = 1$ no other such monomial exists. Thus

$$y^q(z(y+z)(y^3 + \alpha y z^i + z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $3 \le i \le 4$ and $j = 1$.

Setting $r = 0$ gives us monomials in the expansion of

$$y^q(z(y+z)(y^3 + \alpha y z^i))^{\frac{q+1}{2}}$$

and we know from above that

$$y^q(z(y+z)(y^3 + \alpha y z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $i = 3$. Set $t = \frac{q+1}{2}$ and $k = 0$ and suppose s is such that

$$y^{\frac{5q+3}{2} - 2s} z^{q+1+is}$$

is not contained in (y^{2q}, z^{2q}). Suppose there is another monomial of this form. Then there exists a triplet (u, v, w) such that

$$-\frac{q+1}{2} - 2s = (m-3)u + (k-2)v - w \text{ and } \frac{q+1}{2} + is = (i+j+n)u + (i+l)v + w.$$

For $i = 3$ and $j \ge 2$, $(m-1)i + 2j + 2n > 0$ and $2 - i < 0$ which implies $v = 0$ and $w = \frac{q+1}{2}$. Plugging $u = 0$ and $w = \frac{q+1}{2}$ into either of the above equations yields

$v = s$ and $k = l = 0$. Thus in the case $i = 3$ and $j \geq 2$ no other such monomial exists. Thus

$$y^q(z(y+z)(y^3 + \alpha y z^i + z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $i = 3$ and $j \geq 2$.

The monomials in the expansion of

$$z^q(z(y+z)(y^3 + \alpha y z^i + \beta z^{i+j}))^{\frac{q+1}{2}}$$

are in the form

$$y^{2(q+1)+(m-3)r+(k-2)s-t} z^{\frac{3q+1}{2}+(i+j+r)r+(i+l)s+t}.$$

Setting $t = \frac{q-3}{2}$ and $s = r = 0$ yields the monomial

$$y^{\frac{3q+7}{2}} z^{2q-1}.$$

As in D.I and D.II α and β will not contribute another monomial of this form. Thus

$$z^q(z(y+z)(y^3 + \alpha y z^i + \beta z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

If $j \geq i$, Lemma 3.33 implies that

$$k[[x,y,z]]/(x^2 - z(y+z)(y^3 + \alpha y z^i + \beta z^j)) \cong k[[x,y,z]]/(x^2 - z(y+z)(y^3 + y z^i))$$

which is case 5) for $i = 3$. Thus

$$a = z(y+z)(y^3 + \alpha y z^i + \beta z^{i+j})$$

is a unique isomorphism class for $i = 3$ and $1 \leq j < 3$ or $i = 4, j = 1$. This gives case 6).

E) If the initial form is y^3z^2, then by Lemma A.15 $y^3z^2 + b$ where $b \in (y, z)^6$ can be rewritten after some change of variables as $(y^3 + az^3)(z^2 + cy^r)$ where $a \in (y, z) \cap k[[y, z]]$ and c is a unit. By the Weierstrass Preparation theorem $y^3 + az^3$ can be replaced by $u(y, z)(y^3 + a(z)y + b(z))$ where $u(y, z)$ is a unit and $a(z), b(z) \in k[[z]]$. To determine the distinct isomorphism classes which stem from this form we break this situation down into the three cases: I) $a(z) = 0$, II) $b(z) = 0$ and III) $a(z) \neq 0$ and $b(z) \neq 0$. Or in otherwords,

I) $(y^3 + \alpha z^i)(z^2 + cy^r)$, where α is a unit in $k[[z]]$,

II) $(y^3 + \alpha y z^i)(z^2 + cy^r)$, where α is a unit in $k[[z]]$,

III) $(y^3 + \alpha y z^i + \beta z^j)(z^2 + cy^r)$, where α and β are units in $k[[z]]$.

E.I) By Remark 3.24 we need only check that

$$y^q((y^3 + \alpha z^i)(z^2 + cy^r))^{\frac{q+1}{2}} \text{ and } z^q((y^3 + \alpha z^i)(z^2 + cy^r))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

as long as $4 \leq i < 6$.

For now assume $\alpha, c \in k$. The monomials in the expansion of

$$y^q((y^3 + \alpha z^i)(z^2 + cy^r))^{\frac{q+1}{2}} \text{ and } z^q((y^3 + \alpha z^i)(z^2 + cy^r))^{\frac{q+1}{2}}$$

are in the form

$$y^{\frac{5q+3}{2} - 3r + rs} z^{q+1+ir-2s} \text{ and } y^{\frac{3(q+1)}{2} - 3r + rs} z^{2q+1+ir-2s},$$

for $0 \leq r, s \leq \frac{q+1}{2}$. Suppose $s = 0$ and $\frac{5q+3}{2} - 3r < 2q$ in the first expression above. By Lemma 3.28 there exists an r such that $\binom{\frac{q+1}{2}}{r} \neq 0$. Then $r > \frac{q+3}{6}$ which implies $q + 1 + ir > \frac{(i+6)q+(3i+6)}{6}$. Suppose $\frac{(i+3)q+(3i+3)}{6} > 2q$ for all q. Then $(i+6)q + 3i + 6 > 12q$ or $3i + 3 > (6-i)q$. For $i < 6$ we can choose $q > \frac{3i+6}{6-i}$. Thus

$$y^q((y^3 + \alpha z^i)(z^2 + cy^r))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Taking $r = 0$ and $s = 1$ in the second expression above, the monomial we obtain is

$$y^{\frac{3(q+1)}{2}+r} z^{2q-1}$$

which is clearly not in (y^{2q}, z^{2q}) for $q > 2n + 3$.

Suppose $\alpha, c \notin k$, then we can write $\alpha = \Sigma_{j,k \geq 0} \alpha_{jk} y^j z^k$ and $c = \Sigma_{l,m \geq 0} \alpha_{lm} y^l z^m$. Suppose α or c contributes a monomial of the form

$$y^{\frac{3(q+1)}{2}-3r} z^{q+1+ir}.$$

In otherwords for some pair (t, u),

$$y^{\frac{3(q+1)}{2}+(j-3)t+(r+l)u} z^{q+1+(i+k)t+(m-2)u} = y^{\frac{3(q+1)}{2}-3r} z^{q+1+ir}.$$

This equality yields the following two equations

$$(j-3)t + (n+l)u = -3r \text{ and } (i+k)t + (m-2)u = ir.$$

Since $n \geq 3$ and $i \geq 4$, $(n+l)i + 3m - 6 > 0$ implies $u = 0$. Plugging $u = 0$ back into either of the above equations implies $j = k = 0$ and $t = r$. Thus no such monomial exists which implies

$$y^q((y^3+\alpha z^i)(z^2+cy^r))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Similarly, assume there exists some monomial equal to

$$y^{\frac{3(q+1)}{2}+r} z^{2q-1}.$$

In other words,

$$y^{\frac{3(q+1)}{2}+r} z^{2q-1} = y^{\frac{3(q+1)}{2}+(j-3)r+(r+l)s} z^{2q+1+(i+k)r+(m-2)s}.$$

This equality yields the following two equations:

$$n = (j-3)r + (n+l)s \text{ and } -2 = (i+k)r + (m-2)s.$$

Note that $2(j-3)+n(i+k) > 0$ since $i \geq 4$ and $n \geq 3$ which implies that $r = 0$. Hence, α, c contributes no such monomial of the form

$$y^{\frac{3(q+1)}{2}} z^{2q-1}$$

and

$$z^q((z^2+cy^r)(y^3+\alpha z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

This gives us 7).

E.II) By Remark 3.24, we need only check that

$$y^q((y^3+\alpha yz^i)(z^2+cy^r))^{\frac{q+1}{2}} \text{ and } z^q((y^3+\alpha yz^i)(z^2+cy^r))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

as long as $3 \leq i < 4$.

For now assume $\alpha, c \in k$. The monomials in the expansion of

$$y^q((y^3 + \alpha y z^i)(z^2 + cy^r))^{\frac{q+1}{2}} \text{ and } z^q((y^3 + \alpha y z^i)(z^2 + cy^r))^{\frac{q+1}{2}}$$

are in the form

$$y^{\frac{5q+3}{2} - 2r + rs} z^{q+1+ir-2s} \text{ and } y^{\frac{3(q+1)}{2} - 2r + rs} z^{2q+1+ir-2s},$$

for $0 \leq r, s \leq \frac{q+1}{2}$. Suppose $s = 0$ and $\frac{3(q+1)}{2} - 2r < q$ in the first expression above. By Lemma 3.27 there exists an r such that $\binom{\frac{q+1}{2}}{r} \neq 0$. Then $r > \frac{q+3}{4}$ which implies $q + 1 + ir > \frac{(i+2)q+(3i+2)}{4}$. Suppose $\frac{(i+4)q+(3i+4)}{4} > 2q$ for all q. Then $(i+4)q + 3i + 4 > 8q$ or $3i + 2 > (4 - i)q$. For $i < 4$ we can choose $q > \frac{3i+4}{4-i}$. Thus

$$y^q((y^3 + \alpha y z^i)(z^2 + cy^r))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Taking $r = 0$ and $s = 1$ in the second expression, the monomial we obtain is

$$y^{\frac{3(q+1)}{2} + r} z^{q-1}$$

which is clearly not in (y^{2q}, z^q) for $2n + 3 < q$.

Suppose $\alpha, c \notin k$, then we can write $\alpha = \Sigma_{j,k \geq 0} \alpha_k z^k$ and $c = \Sigma_{l,m \geq 0} c_{lm} y^l z^m$. Suppose α or c contribute a monomial of the form

$$y^{\frac{3(q+1)}{2} - 2r} z^{q+1+ir}.$$

In otherwords for some pair (t, u),

$$y^{\frac{3(q+1)}{2} + (j-2)t + (r+l)u} z^{q+1+(i+k)t+(m-2)u} = y^{\frac{3(q+1)}{2} - 2r} z^{q+1+ir}.$$

This equality yields the following two equations

$$(j-2)t+(n+l)u=-2r \text{ and } (i+k)t+(m-2)u=ir.$$

Thus no such monomial exists and

$$y^q((y^3+\alpha yz^i)(z^2+cy^r))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Similarly, assume there exists some monomial equal to

$$y^{\frac{3(q+1)}{2}+r}z^{2q-1}.$$

In other words,

$$y^{\frac{3(q+1)}{2}+r}z^{2q-1} = y^{\frac{3(q+1)}{2}+(j-3)r+(r+l)s}z^{2q+1+(i+k)r+(m-2)s}.$$

This equality yields the following two equations:

$$n=(j-3)r+(n+l)s \text{ and } -2=(i+k)r+(m-2)s.$$

Hence, α, c contributes no such monomial of the form

$$y^{\frac{3(q+1)}{2}}z^{2q-1}$$

and

$$z^q((z^2+cy^r)(y^3+\alpha yz^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

This gives us 8).

E.III) We need only consider the case when $i<j$ since Lemma 3.31 implies that if $i \geq j$ then

$$k[[x,y,z]]/(x^2-(z^2+cy^r)(y^3+\alpha yz^i+\beta z^j)) \cong k[[x,y,z]]/(x^2-(z^2+\alpha y^r)(y^3+z^j))$$

which was case 7) for $4 \leq j < 6$. For simplicity replace j by $i + j$ where $j \geq 1$. Thus we need only consider when

$$a = (z^2 + cy^r)(y^3 + \alpha y z^i + \beta z^{i+j})$$

for $i = 3$ and $j \geq 1$ or $i = 4, j = 1$.

By Remark 3.24, we show that the test ideal must be the maximal ideal by looking at the monomials in the expansion of

$$y^q((z^2 + cy^r)(y^3 + \alpha y z^i + \beta z^{i+j}))^{\frac{q+1}{2}}$$

and

$$z^q((z^2 + cy^r)(y^3 + \alpha y z^i + \beta z^{i+j}))^{\frac{q+1}{2}}.$$

Taking into account that

$$\alpha = \Sigma_{k,l \geq 0} \alpha_{kl} y^k z^l, \beta = \Sigma_{k,l \geq 0} \beta_{kl} y^k z^l \text{ and } c = \Sigma_{k,l \geq 0} c_{kl} y^k z^l,$$

the monomials in the expansion of

$$y^q((z^2 + cy^r)(y^3 + \alpha y z^i + \beta z^{i+j}))^{\frac{q+1}{2}}$$

are in the form

$$y^{\frac{5q+3}{2}(k_1-3)r(k_2-2)s+(r+k_3)t} z^{q+1+(i+j+l_1)r+(i+l_2)s+(l_3-2)t}.$$

Setting $s = 0$ gives us monomials in the expansion of

$$y^q((z^2 + cy^r)(y^3 + \beta z^{i+j}))^{\frac{q+1}{2}}$$

and we know from above that

$$y^q((z^2+cy^r)(y^3+\beta z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $4 \le i+j \le 5$, thus as long as there is no other monomial of the form

$$y^{\frac{5q+3}{2}-3r} z^{q+1+(i+j)r} \notin (y^{2q}, z^{2q})$$

then

$$y^q((z^2+cy^r)(y^3+\alpha y z^i+\beta z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Suppose not, then for some triplet (u, v, w),

$-3r = (k_1-3)u+(k_2-2)v+(n+k_3)w$ and $(i+j)r = (i+j+l_1)u+(i+l_2))v+(l_3-2)w$.

For $3 \le i \le 4$ and $j = 1$, $(1+k_2)i+3l_2-2j, (n+k_3)(i+j)+3l_3-6 > 0$ implies $v = w = 0$. But plugging $v = w = 0$ into either of the above equations implies $r = u$. Thus no such monomial exists for $3 \le i \le 4$ and $j = 1$.

Setting $r = 0$ gives us monomials in the expansion of

$$y^q((z^2+cy^r)(y^3+\alpha z^i))^{\frac{q+1}{2}}$$

and we know from above that

$$y^q((z^2+cy^r)(y^3+\alpha z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $i = 3$, thus as long as there is no other monomial of the form

$$y^{\frac{5q+3}{2}-2s} z^{q+1+is} \notin (y^{2q}, z^{2q})$$

then

$$y^q((z^2+cy^r)(y^3+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Suppose not, then for some triplet (u, v, w),

$-2s = (k_1-3)u+(k_2-2)v+(n+k_3)w$ and $ir = (i+j+l_1)u+(i+l_2))v+(l_3-2)w$.

For $i=3$ and $j \geq 2$, $2(j+l_1)+(k_1-1)i, (n+k_3)i+2l_3-4 > 0$ implies $u = w = 0$. But plugging $u = w = 0$ into either of the above equations implies $s = v$. Thus no such monomial exists for $i = 3$ and $j \geq 2$.

The monomials in the expansion of

$$z^q((z^2+cy^r)(y^3+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}}$$

are in the form

$$y^{\frac{3(q+1)}{2}(k_1-3)r(k_2-2)s+(r+k_3)t}z^{2q+1+(i+j+l_1)r+(i+l_2)s+(l_3-2)t}.$$

Setting $r = s = k_3 = l_3 = 0$ and $t = 1$ the monomial we obtain is

$$y^{\frac{3(q+1)}{2}+r}z^{2q-1}.$$

As in E.I and E.II c, α and β will not contribute another monomial of this form. Thus

$$z^q((z^2+cy^r)(y^3+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

This gives case 9).

F) If the initial form is y^4z, then by Lemma A.16, y^4z+b where $b \in (y,z)^6$ can be rewritten after some change of variables as $((y^2+cz^i)^2+dz^4)z$ where c is either equal to 0 or 1 and $d \in (y,z)$. Suppose first that $c=0$. Since $d \in (y,z)$ it must be in the form $a(y,z)y + b(y,z)z$. To analyze the isomorphism classes generated from this form we must look at the following three cases:

I) $z(y^4 + \alpha z^i)$, where α is a unit in $k[[y,z]]$,

II) $z(y^4 + \alpha y z^i)$, where α is a unit in $k[[y,z]]$,

III) $(y^4 + \alpha y z^i + \beta z^j)$, where α and β are units in $k[[y,z]]$.

F.I) As long as $\alpha \in k$, $x^2 - a = x^2 - z(y^4 + \alpha z^i)$ is a quasihomogeneous polynomial with $\deg(x) = 2i+2$, $\deg(y) = i$ and $\deg(z) = 4$. By Corollary 3.23 $\tau = \mathfrak{m}$ if and only if $\deg(x) = 2i+2 \geq i+4 = \deg(y)+\deg(z)$, $\deg(xy) = 3i+2 < 2i+8 = \deg(y^2)+\deg(z^2)$ and $\deg(xz) = 2i+6 < 2i+8 = \deg(y^2)+\deg(z^2)$. In otherwords, $\tau = \mathfrak{m}$ if and only if $2 \leq i < 6$.

By Remark 3.24, we need only show that for $\alpha \notin k$ and $i=5$ that

$$y^q(z(y^4+\alpha z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4+\alpha z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Note

$$\alpha = \Sigma_{k,l\geq 0} y^k z^l.$$

Suppose α contributes another monomial of the form

$$y^{3q+2-4r} z^{\frac{q+1}{2}+ir} \text{ and } y^{2(q+1)-4r} z^{\frac{3q+1}{2}+ir} \notin (y^{2q}, z^{2q}).$$

Then for some nonzero integer $s \neq r$

$$y^{3q+2-4r} z^{\frac{q+1}{2}+ir} = y^{3q+2+(k-4)s} z^{\frac{q+1}{2}+(i+l)s} \text{ or } y^{2(q+1)-4r} z^{\frac{3q+1}{2}+ir} = y^{2(q+1)+(k-4)s} z^{\frac{3q+1}{2}+(i+l)s}.$$

Both equalities imply

$$-4r = (k-4)s \text{ and } ir = (i+l)s.$$

Thus $k = l = 0$ and we have shown there is no other monomial of the form

$$y^{3q+2-4r}z^{\frac{q+1}{2}+ir} \text{ or } y^{2(q+1)-4r}z^{\frac{3q+1}{2}+ir}.$$

Thus

$$y^q(z(y^4+\alpha z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4+\alpha z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $i = 5$.

By Lemma 3.35, there exist an automorphism taking $x^2 - (z(y^4+\alpha z^5))$ to $x^2 - (z(y^4+z^5))$. Hence when $a = z(y^4+z^5)$ then the test ideal of $R = k[[x,y,z]]/(x^2-a)$ is the maximal ideal. Thus we have 10).

F.II) As long as $\alpha \in k$, $x^2 - a = x^2 - z(y^4 + \alpha y z^i)$ is a quasihomogeneous polynomial with $\deg(x) = 4i+3$, $\deg(y) = 2i$ and $\deg(z) = 6$. By Corollary 3.23 $\tau = \mathfrak{m}$ if and only if $\deg(x) = 4i+3 \geq 2i+6 = \deg(y)+\deg(z)$, $\deg(xy) = 6i+3 < 4i+12 = \deg(y^2)+\deg(z^2)$ and $\deg(xz) = 2i+6 < 2i+12 = \deg(y^2)+\deg(z^2)$. In otherwords, $\tau = \mathfrak{m}$ if and only if $3 \leq i < \frac{9}{2}$. When $i = 3$, $a = z(y^4+\alpha y z^3)$ is a product of five independent linear factors. In case A above we have shown that such a force $k[[x,y,z]]/(x^2-a)$ to have test ideal equal to the maximal ideal.

By Remark 3.24, we need only show that for $i = 4$ that

$$y^q(z(y^4+\alpha y z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4+\alpha y z^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Suppose $\alpha \notin k$ contributes another monomial of the form

$$y^{3q+2-3r}z^{\frac{q+1}{2}+ir} \text{ or } y^{2(q+1)-3r}z^{\frac{3q+1}{2}+ir} \notin (y^{2q}, z^{2q}).$$

Then for some nonzero integer $s \neq r$

$$y^{3q+2-3r}z^{\frac{q+1}{2}+ir} = y^{3q+2+(k-3)s}z^{\frac{q+1}{2}+(i+l)s} \text{ or } y^{2(q+1)-3r}z^{\frac{3q+1}{2}+ir} = y^{2(q+1)+(k-3)s}z^{\frac{3q+1}{2}+(i+l)s}.$$

Both equalities imply

$$-3r = (k-3)s \text{ and } ir = (i+l)s.$$

Thus $k = l = 0$ and we have shown there is no other monomial of the form

$$y^{2(q+1)-3r}z^{\frac{3q+1}{2}+ir} \text{ or } y^{2(q+1)-3r}z^{\frac{3q+1}{2}+ir}.$$

Thus

$$y^q(z(y^4+\alpha yz^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4+\alpha yz^i))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $i = 4$.

By Lemma 3.35, there exist an automorphism taking $x^2 - (z(y^4+\alpha yz^4))$ to $x^2 - (z(y^4+yz^4))$. Hence when $a = z(y^4+yz^4)$ then the test ideal of $R = k[[x,y,z]]/(x^2-a)$ is the maximal ideal. Thus we have 11).

F.III) By Remark 3.24, we need only show that for $i = 4$ and $j \geq 1$ that

$$y^q(z(y^4+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Write $\alpha = \Sigma_{k,l\geq 0}\alpha_{kl}y^kz^l$ $\beta = \Sigma_{m,r\geq 0}\beta_{mr}y^mz^r$ where α_{kl} and β_{mr} are elements of k. The monomials in the expansion of

$$y^q(z(y^4+yz^i+\beta z^j))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}}$$

are of the form

$$y^{3q+2+(m-4)r+(k-3)s}z^{\frac{q+1}{2}+(i+j+r)r+(i+l)s}$$

and

$$y^{2(q+1)+(m-4)r+(k-3)s}z^{\frac{3q+1}{2}+(i+j+r)r+(i+l)s}$$

for $0 \leq r \leq \frac{q+1}{2}$ and $0 \leq s \leq \frac{q+1}{2} - r$. Note that if we take $s = 0$ then we are dealing only with elements in the expansion of

$$y^q(z(y^4 + \beta z^{i+j}))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4 + \beta z^{i+j}))^{\frac{q+1}{2}}$$

which we know is not contained in (y^{2q}, z^{2q}) for $i + j = 5$ by F.I. If we take $r = 0$ then we are dealing only with elements in the expansion of

$$y^q(z(y^4 + \alpha y z^i))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4 + \alpha y z^i))^{\frac{q+1}{2}}$$

which we know is not contained in (y^{2q}, z^{2q}) for $i = 4$ by F.II.

As in F.I, for some r there exists a monomial

$$y^{3q+2-4r}z^{\frac{q+1}{2}+(i+j)r} \text{ or } y^{2(q+1)-4r}z^{\frac{3q+1}{2}+(i+j)r} \notin (y^{2q}, z^{2q}).$$

Suppose there exists a pair (t, u) such that

$$y^{3q+2-4r}z^{\frac{q+1}{2}+(i+j)r} = y^{3q+2+(m-4)t+(k-3)u}z^{\frac{q+1}{2}+(i+j+r)t+(i+l)u}$$

or

$$y^{2(q+1)-4r}z^{\frac{3q+1}{2}+(i+j)r} = y^{2(q+1)+(m-4)t+(k-3)u}z^{\frac{3q+1}{2}+(i+j+r)t+(i+l)u}.$$

Both equalities imply the following system of equations:

$$-4r = (m-4)t + (k-3)u \text{ and } (i+j)r = (i+j+n)t + (i+l)u.$$

Then we see that $i(k+1)-3j+4l>0$ for $i=4$ and $j=1$ and this implies $(r,0)=(t,u)$ and

$$y^q(z(y^4+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

As in F.II, for some s there exists a monomial

$$y^{3q+2-3s}z^{\frac{q+1}{2}+is} \text{ or } y^{2(q+1)-3s}z^{\frac{3q+1}{2}+is} \notin (y^{2q}, z^{2q}).$$

Suppose there exists a pair (t,u) such that

$$-3s=(m-4)t+(k-3)u \text{ and } is=(i+j+n)t+(i+l)u.$$

Then we see that $i(m-1)+3(j+n)>0$ for $i=4$ and $j\geq 2$ and this implies $(0,s)=(t,u)$

$$y^q(z(y^4+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}} \text{ and } z^q(z(y^4+\alpha yz^i+\beta z^{i+j}))^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Thus we have 12).

Now consider $z((y^2+z^i)^2+dz^4)$ where $d\in(y,z)$. Since $d\in(y,z)$ it must be in the form $a(y,z)y+b(y,z)z$. To analyze the isomorphism classes generated from this form we must look at the following three cases where $n>i$:

IV) $(y^2+z^i)^2+\alpha z^r$, where c and α are units in $k[[y,z]]$,

V) $(y^2+z^i)^2+\alpha yz^r$, where c and α are units in $k[[y,z]]$,

VI) $(y^2+z^i)^2+\alpha yz^r+\beta z^m$, where c, α and β are units in $k[[y,z]]$.

F.IV) To determine which i force

$$y^q(z(y^2+z^i)^2+\alpha yz^r)^{\frac{q+1}{2}} \text{ and } z^q(z(y^2+z^i)^2+\alpha yz^r)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

we need to look at the monomials in the expansion of

$$y^q(z(y^2+z^i)^2+\alpha z^r)^{\frac{q+1}{2}} \text{ and } z^q(z(y^2+z^i)^2+\alpha z^r)^{\frac{q+1}{2}}$$

by Remark 3.24. As long as $\alpha \in k$ the monomials in the expansion of

$$y^q(z(y^2+z^i)^2+\alpha z^r)^{\frac{q+1}{2}} \text{ and } z^q(z(y^2+z^i)^2+\alpha z^r)^{\frac{q+1}{2}}$$

have the form

$$y^{3q+2-4r-2s}z^{\frac{q+1}{2}+rr+is} \text{ and } y^{2(q+1)-4r-2s}z^{\frac{3q+1}{2}+rr+is}$$

for $0 \leq r \leq \frac{q+1}{2}$ and $0 \leq s \leq q+1-2r$. If we set $r=1$ and $t=\frac{q+1}{2}$ in the first expression above, the monomial we obtain is

$$y^{2q-3}z^{\frac{(i+1)(q+1)}{2}+r}$$

which has nonzero coefficient modulo p by Lemma 3.29. If $i>2$ then

$$y^{2q-3}z^{\frac{(i+1)(q+1)}{2}+r}$$

is contained in (y^{2q}, z^{2q}). Setting $r=1$ and $t=0$ in the second expression, the monomial we obtain is

$$y^{2q-1}z^{\frac{3q+1}{2}+r}.$$

As long as $i=2$, each are monomials that are not contained in (y^{2q}, z^{2q}) for $q>2n+3$. But we were assuming that $i \geq 3$. When $i=2$ this case reduces to C).

Similarly we show that F.V) and F.VI) reduce to case C).

G) If the initial form is y^5, then by Lemma A.17, $y^5 + b$ where $b \in (y,z)^6$ can be rewritten after some change of variables as

$$y^5 + a(z)y^3z^3 + b(z)y^2z^4 + c(z)yz^5 + d(z)z^6$$

where $a(z), b(z), c(z), d(z) \in k[[z]]$. Classification of the isomorphism classes breaks down into the following cases for α, β, γ and δ units:

I) $y^5 + \alpha z^i$,

II) $y^5 + \alpha y z^i$,

III) $y^5 + \alpha y z^i + \beta z^j$,

IV) $y^5 + \alpha y^2 z^i + \beta z^j$,

V) $y^5 + \alpha y^3 z^i + \beta z^j$,

VI) $y^5 + \alpha y^2 z^i + \beta y z^j$,

VII) $y^5 + \alpha y^3 z^i + \beta y z^j$,

VIII) $y^5 + \alpha y^2 z^i + \beta y z^j + \gamma z^k$,

IX) $y^5 + \alpha y^3 z^i + \beta y z^j + \gamma z^k$,

X) $y^5 + \alpha y^3 z^i + \beta y^2 z^j + \gamma z^k$,

XI) $y^5 + \alpha y^3 z^i + \beta y^2 z^j + \gamma y z^k + \delta z^l$.

G.I) As long as $\alpha \in k$, $x^2 - a = x^2 - (y^5 + \alpha z^i)$ is a quasihomogeneous polynomial with $\deg(x) = 5i$, $\deg(y) = 2i$ and $\deg(z) = 10$. By Corollary 3.23 $\tau = \mathfrak{m}$ if and only if $\deg(x) = 5i \geq 2i + 10 = \deg(y) + \deg(z)$, $\deg(xy) = 7i < 4i + 20 = \deg(y^2) + \deg(z^2)$ and $\deg(xz) = 5i + 10 < 4i + 20 = \deg(y^2) + \deg(z^2)$. In otherwords, $\tau = \mathfrak{m}$ if and only if $10/3 \leq i < 20/3$.

When $\alpha \notin k$ by Remark 3.24, we look at the monomials in the expansion of

$$y^q(y^5 + \alpha z^i)^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha z^i)^{\frac{q+1}{2}}.$$

Write $\alpha = \Sigma_{k,l \geq 0} \alpha_{kl} y^k z^l$. The monomials are of the form

$$y^{\frac{7q+5}{2}+(k-5)r} z^{(i+l)r} \text{ or } y^{\frac{5(q+1)}{2}+(k-5)r} z^{q+(i+l)r}.$$

Suppose α contributes a monomial of the form

$$y^{\frac{7q+5}{2}-5r} z^{ir} \text{ or } y^{\frac{5(q+1)}{2}-5r} z^{q+ir}$$

not contained in (y^{2q}, z^{2q}) coming from the expansion of

$$y^q(y^5 + \alpha_{00} z^i)^{\frac{q+1}{2}} \text{ or } z^q(y^5 + \alpha_{00} z^i)^{\frac{q+1}{2}}.$$

Then there exists an integer $s \neq 0$ such that

$$y^{\frac{7q+5}{2}-5r} z^{ir} = y^{\frac{7q+5}{2}+(k-5)s} z^{(i+l)s}$$

or

$$y^{\frac{5(q+1)}{2}-5r} z^{q+ir} = y^{\frac{5(q+1)}{2}+(k-5)s} z^{q+(i+l)s}.$$

Both of these equalities give the following two equations

$$-5r = (k-5)s \text{ and } ir = (i+l)s.$$

Thus α does not contribute a monomial of the form

$$y^{\frac{7q+5}{2}-5r} z^{ir} \text{ or } y^{\frac{5(q+1)}{2}-5r} z^{q+ir}.$$

Thus both

$$y^q(y^5+\alpha z^i)^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha z^i)^{\frac{q+1}{2}}$$

are not in $y^{2q}, z^{2q})$.

By Lemma 3.39 there exists an automorphism of $k[[x,y,z]]$ taking $y^5+\alpha z^i$ to y^5+z^i. Thus $k[[x,y,z]]/(x^2-(y^5+z^i))$ has test ideal equal to the maximal ideal. Thus we have case 13).

G.II) As long as $\alpha \in k$, $x^2-a = x^2-(y^5+\alpha y z^i)$ is a quasihomogeneous polynomial with $\deg(x) = 5i$, $\deg(y) = 2i$ and $\deg(z) = 8$. By Corollary 3.23 $\tau = \mathfrak{m}$ if and only if $\deg(x) = 5i \geq 2i+8 = \deg(y)+\deg(z)$, $\deg(xy) = 7i < 4i+16 = \deg(y^2)+\deg(z^2)$ and $\deg(xz) = 5i+8 < 4i+16 = \deg(y^2)+\deg(z^2)$. In otherwords, $\tau = \mathfrak{m}$ if and only if $8/3 \leq i < 16/3$.

When $\alpha \notin k$ by Remark 3.24, we look at the monomials in the expansion of

$$y^q(y^5+\alpha y z^i)^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha y z^i)^{\frac{q+1}{2}}.$$

Write $\alpha = \Sigma_{k,l\geq 0}\alpha_{kl}y^k z^l$. The monomials have the form

$$y^{\frac{7q+5}{2}+(k-4)r}z^{(i+l)r} \text{ and } y^{\frac{5(q+1)}{2}+(k-4)r}z^{q+(i+l)r},$$

for $0 \leq r \leq \frac{q+1}{2}$.

Suppose α contributes a monomial of the form

$$y^{\frac{7q+5}{2}-4r}z^{ir} \text{ or } y^{\frac{5(q+1)}{2}-4r}z^{q+ir}.$$

Then there exists an integer $s \neq 0$ such that

$$y^{\frac{7q+5}{2}-4r}z^{ir} = y^{\frac{7q+5}{2}+(k-4)s}z^{(i+l)s}$$

or

$$y^{\frac{5(q+1)}{2}-4r}z^{q+ir} = y^{\frac{5(q+1)}{2}+(k-4)s}z^{q+(i+l)s}.$$

Both of these equalities give the following system of equations

$$-4r = (k-4)s \text{ and } ir = (i+l)s.$$

Thus α does not contribute a monomial of the form

$$y^{\frac{7q+5}{2}-4r}z^{ir} \text{ or } y^{\frac{5(q+1)}{2}-4r}z^{q+ir}.$$

Thus both

$$y^q(y^5+\alpha yz^i)^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha yz^i)^{\frac{q+1}{2}}$$

are not in (y^{2q}, z^{2q}).

By Lemma 3.39 there exists an automorphism of $k[[x,y,z]]$ taking $y^5+\alpha yz^i$ to y^5+yz^i. Thus $k[[x,y,z]]/(x^2-(y^5+yz^i))$ has test ideal equal to the maximal ideal. Thus we have case 14).

G.III) Note if $i \geq j$ then $\beta+\alpha yz^{i-j}$ is a unit; therefore, in this case $y^5+\alpha yz^i+\beta z^j$ is in the same isomorphism class as y^5+z^j by case G.I). So we may assume that $i < j$, and replace j by $i+j$ where $j \geq 1$. By Lemma 3.40, $j < i$ otherwise $y^5+\alpha yz^i+\beta z^{i+j}$ is in the same isomorphism class as y^5+yz^i which is case G.II).

Suppose

$$y^q(y^5+\alpha yz^i+\beta z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha yz^i+\beta z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Then some monomials in the expansion of

$$y^q(y^5+yz^i+\beta z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5+yz^i+\beta z^{i+j})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). The monomials have the form

$$y^{\frac{7q+5}{2}-4r-5s} z^{ir+(i+j+k)s} \text{ and } y^{\frac{5(q+1)}{2}-4r-5s} z^{q+(i+k)r+(i+j+l)s},$$

for $0 \le s \le \frac{q+1}{2}$ and $0 \le r \le \frac{q+1}{2} - s$ and where k and l are the powers of z which α and β contribute.

When $s = 0$ and $j > 1$ then the monomials occuring are just those from the expansion of $y^q(y^5 + \alpha y z^i)^{\frac{q+1}{2}}$ and $z^q(y^5 + \alpha y z^i)^{\frac{q+1}{2}}$. By case G.II) we know that i must be 5. Suppose that α and β contribute a monomial either of the form

$$y^{\frac{7q+5}{2}-4r} z^{5r} \text{ or } y^{\frac{5(q+1)}{2}-4r} z^{q+5r} \notin (y^{2q}, z^{2q}).$$

Then there will exist some pair (t, u) such that

$$y^{\frac{7q+5}{2}-4r} z^{5r} = y^{\frac{7q+5}{2}-4t-5u} z^{(5+k)t+(5+j+l)u}$$

or

$$y^{\frac{5(q+1)}{2}-4r} z^{q+5r} = y^{\frac{5(q+1)}{2}-4t-5u} z^{q+(5+k)t+(5+j+l)u}.$$

Both equalities yield the following system of equations:

$$-4r = -4t - 5u \text{ and } 5r = (5+k)t + (5+j+l)u.$$

For $j > 1$, $(4(5+j+k) - 25) > 0$, thus $u = 0$ and $t = r$.

When $r = 0$ and $j = 1$ then the monomials occuring are just those from the expansion of $y^q(y^5 + \beta z^{i+j})^{\frac{q+1}{2}}$ and $z^q(y^5 + \beta z^{i+j})^{\frac{q+1}{2}}$. By case G.I) we know that $i + j$ must be 6. Suppose that α and β contribute a monomial either of the form

$$y^{\frac{7q+5}{2}-5s} z^{6s} \text{ or } y^{\frac{5(q+1)}{2}-5s} z^{q+6s} \notin (y^{2q}, z^{2q})$$

Then there will exist some pair (t, u) such that

$$y^{\frac{7q+5}{2}-5s}z^{6s} = y^{\frac{7q+5}{2}-4t-5u}z^{(5+k)t+(6+l)u}$$

or

$$y^{\frac{5(q+1)}{2}-5s}z^{q+6s} = y^{\frac{5(q+1)}{2}-4t-5u}z^{q+(5+k)t+(6+l)u}.$$

Both equalities yield the following system of equations:

$$-5s = -4t - 5u \text{ and } 6s = (5+k)t + (6+l)u.$$

Thus for $j = 1$, $5lu + (5(5+k) - 24 > 0$, thus $t = 0$ and $u = s$.

Both arguments above imply that

$$y^q(y^5 + \alpha y z^i + \beta z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y z^i + \beta z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence by Remark 3.24, we have case 15).

G.IV) Note if $i \geq j$ then $\beta + \alpha y^2 z^{i-j}$ is a unit with a jth root; therefore, in this case $y^5 + \alpha y^2 z^i + \beta z^j$ is in the same isomorphism class as $y^5 + \beta z^j$ and this is case G.I). So we may assume that $i < j$, or that we can replace j by $i + j$. Suppose

$$y^q(y^5 + \alpha y^2 z^i + \beta z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^2 z^i + \beta z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Then some monomials in the expansion of

$$y^q(y^5 + \alpha y^2 z^i + \beta z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^2 z^i + \beta z^{i+j})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). The monomials have the form

$$y^{\frac{7q+5}{2}-3r-5s}z^{(i+k)r+(i+j+l)s} \text{ and } y^{\frac{5(q+1)}{2}-3r-5s}z^{q+(i+k)r+(i+j+l)s},$$

for $0 \leq s \leq \frac{q+1}{2}$ and $0 \leq \frac{q+1}{2} - s$ and where k and l are the powers of z which α and β contribute.

When $r = 0$ then the monomials occuring are just those from the expansion of $y^q(y^5 + \beta z^j)^{\frac{q+1}{2}}$ and $z^q(y^5 + \beta z^j)^{\frac{q+1}{2}}$. By case G.I) we know that $i + j$ must be 6. Suppose that α or β contributes a monomial either of the form

$$y^{\frac{7q+5}{2} - 5s} z^{6s} \text{ or } y^{\frac{5(q+1)}{2} - 5s} z^{q+6s} \notin (y^{2q}, z^{2q}).$$

Then there will exist some pair (t, u) such that

$$y^{\frac{7q+5}{2} - 5s} z^{6s} = y^{\frac{7q+5}{2} - 3t - 5u} z^{(6-j+k)t + (6+l)u}$$

or

$$y^{\frac{5(q+1)}{2} - 5s} z^{q+6s} = y^{\frac{5(q+1)}{2} - 3t - 5u} z^{q + (6-j+k)t + (6+l)u}.$$

Both equalities yield the following system of equations:

$$-5s = -3t - 5u \text{ and } 6s = (6 - j + k)t + (6 + l)u.$$

For $1 \leq j \leq 2$, $(5(6 - j + k) - 18) > 0$, thus $t = 0$ and $u = s$. Thus

$$y^q(y^5 + \alpha y^2 z^{6-j} + \beta z^6)^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^2 z^{6-j} + \beta z^6)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $1 \leq j \leq 2$. Hence by Remark 3.24 we have case 16).

G.V) Note if $i \geq j$ then $\beta + \alpha y^3 z^{i-j}$ is a unit with a jth root; therefore, in this case $y^5 + \alpha y^3 z^i + \beta z^j$ is in the same isomorphism class as $y^5 + z^j$ and this is case G.I). So we may assume that $i < j$, and we replace j by $i + j$. Suppose

$$y^q(y^5 + \alpha y^3 z^i + z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^3 z^i + z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Then some monomials in the expansion of

$$y^q(y^5+\alpha y^3z^i+\beta z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha y^3z^i+\beta z^{i+j})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). The monomials have the form

$$y^{\frac{7q+7}{2}-2r-5s}z^{(i+k)r+(i+j+l)s} \text{ and } y^{\frac{5(q+1)}{2}-2r-5s}z^{q+(i+k)r+(i+j+l)s},$$

for $0 \leq s \leq \frac{q+1}{2}$ and $0 \leq r \leq \frac{q+1}{2} - s$ and where k and l are the powers of z which α and β contribute.

When $r = 0$ then the monomials occuring are just those from the expansion of $y^q(y^5+z^j)^{\frac{q+1}{2}}$ and $z^q(y^5+z^j)^{\frac{q+1}{2}}$. By case G.I) we know that i must be 6. Suppose that α and β contribute a monomial of the form

$$y^{\frac{7q+5}{2}-5s}z^{6s} \text{ or } y^{\frac{5(q+1)}{2}-5s}z^{q+6s} \notin (y^{2q}, z^{2q}).$$

Then there will exist some pair (t, u) such that

$$y^{\frac{7q+5}{2}-5s}z^{6s} = y^{\frac{7q+5}{2}-2t-5u}z^{(6-j+k)t+(6+l)u}$$

or

$$y^{\frac{5(q+1)}{2}-5s}z^{q+6s} = y^{\frac{5(q+1)}{2}-2t-5u}z^{q+(6-j+k)t+(6+l)u}.$$

Both equalities yield the following system of equations:

$$-5s = -2t - 5u \text{ and } 6s = (6-j+k)t + (6+l)u.$$

For $1 \leq j \leq 3$, $(5(6-j+k) - 12) > 0$, thus $t = 0$ and $u = s$. Thus

$$y^q(y^5+\alpha y^3z^i+z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha y^3z^i+z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence by Remark 3.24, we have case 17).

G.VI) Note if $i \geq j$ then $\beta + \alpha y z^{i-j}$ is a unit with a jth root; therefore, in this case $y^5 + \alpha y^2 z^i + \beta y z^j$ is in the same isomorphism class as $y^5 + y z^j$ and this is case G.II). So we may assume that $i < j$, and replace j by $i + j$. Suppose

$$y^q(y^5 + \alpha y^2 z^i + \beta y z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^2 z^i + \beta y z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Then some monomials in the expansion of

$$y^q(y^5 + \alpha y^2 z^i + \beta y z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^2 z^i + \beta y z^{i+j})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). The monomials have the form

$$y^{\frac{7q+5}{2}-3r-4s} z^{(i+k)r+(i+j+l)s} \text{ and } y^{\frac{5(q+1)}{2}-3r-4s} z^{q+(i+k)r+(i+j+l)s},$$

for $0 \leq s \leq \frac{q+1}{2}$ and $0 \leq r \leq \frac{q+1}{2} - s$ and where k and l are the powers of z which α and β contribute.

When $r = 0$ then the monomials occuring are just those from the expansion of $y^q(y^5 + y z^{i+j})^{\frac{q+1}{2}}$ and $z^q(y^5 + y z^{i+j})^{\frac{q+1}{2}}$. By case G.II) we know that $i + j$ must be 5. Suppose that α and β contribute a monomial of the form

$$y^{\frac{7q+5}{2}-4s} z^{5s} \text{ or } y^{\frac{5(q+1)}{2}-4s} z^{q+5s} \notin (y^{2q}, z^{2q}).$$

Then there will exist some pair (t, u) such that

$$y^{\frac{7q+5}{2}-4s} z^{5s} = y^{\frac{7q+5}{2}-3t-4u} z^{(4+k)t+(5+l)u}$$

or

$$y^{\frac{5(q+1)}{2}-4s} z^{q+5s} = y^{\frac{5(q+1)}{2}-3t-4u} z^{q+(4+k)t+(5+l)u}.$$

Both equalities yield the following system of equations:

$$-4s = -3t - 4u \text{ and } 5s = (4+k)t + (5+l)u.$$

Since $(4(4+k) - 15) > 0$, then $t = 0$ and $u = s$. Thus

$$y^q(y^5 + \alpha y^2 z^4 + \beta y z^5)^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^2 z^4 + \beta y z^5)^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Hence by Remark 3.24, we have case 18).

G.VII) Note if $i \geq j$ then $\beta + \alpha y^2 z^{i-j}$ is a unit with a jth root; therefore, in this case $y^5 + \alpha y^3 z^i + \beta y z^j$ is in the same isomorphism class as $y^5 + yz^j$ and this is case G.II). So we may assume that $i < j$, and replace j by $i + j$. Suppose

$$y^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q}).$$

Then some monomials in the expansion of

$$y^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). The monomials have the form

$$y^{\frac{7q+5}{2} - 2r - 4s} z^{(i+k)r + (i+j+l)s} \text{ and } y^{\frac{5(q+1)}{2} - 2r - 4s} z^{q + (i+k)r + (i+j+l)s},$$

for $0 \leq s \leq \frac{q+1}{2}$ and $0 \leq r \leq \frac{q+1}{2} - s$ where k and l are the powers of z which α and β contributes.

When $r = 0$ then the monomials occuring are just those from the expansion of $y^q(y^5 + \beta y z^{i+j})^{\frac{q+1}{2}}$ and $z^q(y^5 + \beta y z^{i+j})^{\frac{q+1}{2}}$. By case G.II) we know that $i + j$ must be 5. Suppose that α or β contributes a monomial of the form

$$y^{\frac{7q+5}{2} - 4s} z^{5s} \text{ or } y^{\frac{5(q+1)}{2} - 4s} z^{q+5s} \notin (y^{2q}, z^{2q}).$$

Then there will exist some pair (t,u) such that

$$y^{\frac{7q+5}{2}-4s}z^{5s}=y^{\frac{7q+5}{2}-2t-4u}z^{(5-j+k)t+(5+l)u}$$

or

$$y^{\frac{5(q+1)}{2}-4s}z^{q+5s}=y^{\frac{5(q+1)}{2}-2t-4u}z^{q+(5-j+k)t+(5+l)u}.$$

Both equalities yield the following system of equations:

$$-4s=-2t-4u \text{ and } 5s=(5-j+k)t+(5+l)u.$$

For $1\le j\le 2$, $(4(5-j+k)-10)>0$, thus $t=0$ and $u=s$. Thus

$$y^q(y^5+\alpha y^3z^i+\beta yz^{i+j})^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha y^3z^i+\beta yz^{i+j})^{\frac{q+1}{2}}\notin(y^{2q},z^{2q}).$$

Hence by Remark 3.24, we have case 19).

G.VIII) Note if $i\ge j\ge k$ then $\gamma+\alpha y^2z^{i-k}+\beta yz^{j-k}$ is a unit with a kth root; therefore, in this case $y^5+\alpha y^2z^i+\beta yz^j+\gamma z^k$ is in the same isomorphism class as $y^5+\gamma z^k$ and this is case G.I). If $i\ge j$ then $\beta+\alpha yz^{i-j}$ is a unit with a jth root; therefore, in this case $y^5+\alpha y^2z^i+\beta yz^j+\gamma z^k$ is in the same isomorphism class as $y^5+\beta yz^j+\gamma z^k$ and this is case G.III). So we may assume that $i<j<k$, and we replace j by $i+j$ and k by $i+j+k$. Suppose

$$y^q(y^5+\alpha y^2z^i+\beta yz^{i+j}+\gamma z^{i+j+k})^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha y^2z^i+\beta yz^{i+j}+\gamma z^{i+j+k})^{\frac{q+1}{2}}$$

are not in (y^{2q},z^{2q}). Then some monomials in the expansion of

$$y^q(y^5+\alpha y^2z^i+\beta yz^{i+j}+\gamma z^{i+j+k})^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha y^2z^i+\beta yz^{i+j}+\gamma z^{i+j+k})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). The monomials have the form

$$y^{\frac{5(q+1)}{2}-3r-4s-5t}z^{(i+l)r+(i+j+m)s+(i+j+k+r)t}$$

and

$$y^{\frac{5(q+1)}{2}-3r-4s-5t}z^{(i+l)r+(i+j+m)s+(i+j+k+r)t},$$

for $0 \leq t \leq \frac{q+1}{2}$, $0 \leq s \leq \frac{q+1}{2} - t$ and $0 \leq r \leq \frac{q+1}{2} - t - s$ and where l, m and n are the powers of z which α, β and γ contribute.

When $r = t = 0$ then the monomials occuring are just those from the expansion of $y^q(y^5 + \beta y z^{i+j})^{\frac{q+1}{2}}$ and $z^q(y^5 + \beta y z^{i+j})^{\frac{q+1}{2}}$. By case G.II) we know that $i + j$ must be 5. Suppose that α, β or γ contribute a monomial of the form

$$y^{\frac{7q+5}{2}-4s}z^{5s} \text{ or } y^{\frac{5(q+1)}{2}-4s}z^{q+5s} \notin (y^{2q}, z^{2q}).$$

Then there will exist some triplet (u, v, w) such that

$$y^{\frac{7q+5}{2}-4s}z^{5s} = y^{\frac{7q+5}{2}-3u-4v-5w}z^{(4+l)u+(5+m)v+(5+k+r)w}$$

or

$$y^{\frac{5(q+1)}{2}-4s}z^{q+5s} = y^{\frac{5(q+1)}{2}-3u-4v-5w}z^{q+(4+l)u+(5+m)v+(5+k+r)w}.$$

Both equalities yields the following system of equations:

$$-4s = -3u - 4v - 5w \text{ and } 5s = (4 + l)u + (5 + m)v + (5 + k + n)w.$$

For $k \geq 2$ $(4(5+k+n) - 25), (4(4+l) - 15) > 0$, thus $u = w = 0$ and $v = s$. Thus

$$y^q(y^5 + \alpha y^2 z^i + \beta y z^{i+j} + \gamma z^{i+j+k})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^2 z^i + \beta y z^{i+j} + \gamma z^{i+j+k})^{\frac{q+1}{2}}$$

are not in (y^{2q}, z^{2q}) for $k \geq 2$. This gives case 22). By Lemma 3.41, $k = 1$ reduces to G.X).

G.IX) Note if $i \geq j \geq k$ then $\gamma + \alpha y^3 z^{i-k} + \beta y z^{j-k}$ is a unit with a kth root; therefore, in this case $y^5 + \alpha y^3 z^i + \beta y z^j + \gamma z^k$ is in the same isomorphism class as $y^5 + \gamma z^k$ and this is case G.I). If $i \geq j$ then $\beta + \alpha y^2 z^{i-j}$ is a unit with a jth root; therefore, in this case $y^5 + \alpha y^3 z^i + \beta y z^j + \gamma z^k$ is in the same isomorphism class as $y^5 + \beta y z^j + \gamma z^k$ and this is case G.III). So we may assume that $i < j < k$, and replace j by $i + j$ and k by $i + j + k$. Suppose

$$y^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j} + \gamma z^{i+j+k})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j} + \gamma z^{i+j+k})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). Then some monomials in the expansion of

$$y^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j} + \gamma z^{i+j+k})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j} + \gamma z^{i+j+k})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). The monomials have the form

$$y^{\frac{7q+5}{2} - 2r - 4s - 5t} z^{(i+l)r + (i+j+m)s + (i+j+k+r)t}$$

and

$$y^{\frac{5(q+1)}{2} - 2r - 4s - 5t} z^{q + (i+l)r + (i+j+m)s + (i+j+k+r)t},$$

for $0 \leq t \leq \frac{q+1}{2}$, $0 \leq s \leq \frac{q+1}{2} - t$ and $0 \leq r \leq \frac{q+1}{2} - t - s$ and where l, m and n are the powers of z which α, β and γ contribute.

When $r = t = 0$ then the monomials occuring are just those from the expansion of $y^q(y^5 + \beta y z^{i+j})^{\frac{q+1}{2}}$ and $z^q(y^5 + \beta y z^{i+j})^{\frac{q+1}{2}}$. By case G.I) we know that $i + j$ must be 5. Suppose that α, β or γ contribute a monomial of the form

$$y^{\frac{7q+5}{2}-4s}z^{5s} \text{ or } y^{\frac{5(q+1)}{2}-4s}z^{q+5s} \notin (y^{2q}, z^{2q}).$$

Then there will exist some triplet (u, v, w) such that

$$y^{\frac{7q+5}{2}-4s}z^{5s} = y^{\frac{7q+5}{2}-2u-4v-5w}z^{(5-j+l)u+(5+m)v+(5+k+r)w}$$

or

$$y^{\frac{5(q+1)}{2}-4s}z^{q+5s} = y^{\frac{5(q+1)}{2}-2u-4v-5w}z^{q+(5-j+l)u+(5+m)v+(5+k+r)w}.$$

Both equalities yield the following system of equations:

$$-4s = -2u - 4v - 5w \text{ and } 5s = (5 - j + l)u + 5v + (5 + k + m)w.$$

For $1 \leq j \leq 2$ and $k \geq 2$ $(4(5+k+m)-25), (4(5-j+l)-10) > 0$, thus $u = w = 0$ and $s = v$. Thus

$$y^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j} + \gamma z^{i+j+k})^{\frac{q+1}{2}} \text{ and } z^q(y^5 + \alpha y^3 z^i + \beta y z^{i+j} + \gamma z^{i+j+k})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). Hence by Remark 3.24, we have case 21). By Lemma 3.41, $k = 1$ reduces to G.X).

G.X) Note if $i \geq j \geq k$ then $\gamma + \alpha y^3 z^i + \beta y^2 z^{j-k}$ is a unit with a kth root; therefore, in this case $y^5 + \alpha y^3 z^i + \beta y^2 z^j + \gamma z^k$ is in the same isomorphism class as $y^5 + \gamma z^k$ and this is case G.I). If $i \geq j$ then $\beta + \alpha y z^{i-j}$ is a unit with a jth root; therefore, in this case $y^5 + \alpha y^3 z^{i-k} + \beta y^2 z^j + \gamma z^k$ is in the same isomorphism class as $y^5 + \beta y^2 z^j + \gamma z^k$ and this is case G.VI). So we may assume that $i < j < k$, and replace j by $i+j$ and k by $i+j+k$. Suppose

$$y^q(y^5+\alpha y^3z^i+\beta y^2z^{i+j}+\gamma z^{i+j+k})^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha y^3z^i+\beta y^2z^{i+j}+\gamma z^{i+j+k})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). Then some monomials in the expansion of

$$y^q(y^5+\alpha y^3z^i+\beta y^2z^{i+j}+z^{i+j+k})^{\frac{q+1}{2}} \text{ and } z^q(y^5+\alpha y^3z^i+\beta y^2z^{i+j}+z^{i+j+k})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). The monomials have the form

$$y^{\frac{7q+5}{2}-2r-3s-5t}z^{(i+l)r+(i+j+m)s+(i+j+k+r)t}$$

and

$$y^{\frac{5(q+1)}{2}-2r-3s-5t}z^{q+(i+l)r+(i+j+m)s+(i+j+k+r)t},$$

for $0 \le t \le \frac{q+1}{2}$, $0 \le s \le \frac{q+1}{2}-t$ and $0 \le r \le \frac{q+1}{2}-t-s$ and where l, m and n are the powers of z which α, β and γ contribute.

When $r=s=0$ then the monomials occuring are just those from the expansion of $y^q(y^5+z^{i+j+k})^{\frac{q+1}{2}}$ and $y^q(y^5+z^{i+j+k})^{\frac{q+1}{2}}$. By case G.I) we know that $i+j+k$ must be 6. Suppose that α, β or γ contributes a monomial of the form

$$y^{\frac{7q+5}{2}-5t}z^{6t} \text{ or } y^{\frac{5(q+1)}{2}-5t}z^{q+6t} \notin (y^{2q}, z^{2q}).$$

Then there will exist some triplet (u, v, w) such that

$$y^{\frac{7q+5}{2}-5t}z^{6t} = y^{\frac{7q+5}{2}-2u-3v-5w}z^{(6-j-k+l)u+(6-k+m)v+(6+r)w}$$

or

$$y^{\frac{5(q+1)}{2}-5t}z^{q+6t} = y^{\frac{5(q+1)}{2}-2u-3v-5w}z^{q+(6-j-k+l)u+(6-k+m)v+(6+r)w}$$

Both equalities yield the following system of equations:

$$-5t = -2u - 3v - 5w \text{ and } 6t = (6 - j - k + l)u + (6 - k + m)v + (6 + n)w.$$

For $1 \le k \le 2$ and $1 \le j \le 2$ where $j + k \le 3$, $(5(6 - k + m) - 18), (5(6 - j - k + l) - 12) > 0$, thus $u = v = 0$ and $w = t$. Hence by Remark 3.24, we have case 20).

G.XI) Note if $i \ge j \ge k \ge l$ then $\delta + \alpha y^3 z^{i-l} + \beta y^2 z^{j-l} + \gamma y z^{k-l}$ is a unit with a lth root; therefore, in this case $y^5 + \alpha y^3 z^i + \beta y^2 z^j + \gamma y z^k + \delta z^l$ is in the same isomorphism class as $y^5 + z^l$ and this is case G.I). If $i \ge j \ge k$ then $\gamma + \alpha y^3 z^{i-k} + \beta y^2 z^{j-k}$ is a unit with a kth root; therefore, in this case $y^5 + \alpha y^2 z^i + \beta y z^j + \gamma y z^k + \delta z^l$ is in the same isomorphism class as $y^5 + \gamma y z^k + \delta z^l$ and this is case G.III). If $i \ge j$ then $\beta + \alpha y z^{i-j}$ is a unit with a kth root; therefore, in this case $y^5 + \alpha y^3 z^i + \beta y^2 z^j + \gamma y z^k + \delta z^l$ is in the same isomorphism class as $y^5 + \beta y^2 z^j + \gamma y z^k + \delta z^l$ and this is case G.X). So we may assume that $i < j < k < l$, and replace j by $i + j$, k by $i + j + k$ and l by $i + j + k + l$. Suppose

$$y^q(y^5 + \alpha y^3 z^i + \beta y^2 z^{i+j} + \gamma y z^{i+j+k} + \delta z^{i+j+k+l})^{\frac{q+1}{2}}$$

and

$$z^q(y^5 + \alpha y^3 z^i + \beta y^2 z^{i+j} + \gamma y z^{i+j+k} + \delta z^{i+j+k+l})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). Then some monomials in the expansion of

$$y^q(y^5 + \alpha y^3 z^i + \beta y^2 z^{i+j} + y z^{i+j+k} + \delta z^{i+j+k+l})^{\frac{q+1}{2}}$$

and

$$z^q(y^5 + \alpha y^3 z^i + \beta y^2 z^{i+j} + y z^{i+j+k} + \delta z^{i+j+k+l})^{\frac{q+1}{2}}$$

are not contained in (y^{2q}, z^{2q}). The monomials have the form

$$y^{\frac{7q+5}{2}-2r-3r-4s-5t} z^{(i+m_1)r+(i+j+m_2)r+(i+j+k+m_3)s+(i+j+k+l+m_4)t}$$

and

$$y^{\frac{5(q+1)}{2}-2r-3r-4s-5t} z^{q+(i+m_1)r+(i+j+m_2)r+(i+j+k+m_3)s+(i+j+k+l+m_4)t},$$

for $0 \leq t \leq \frac{q+1}{2}$, $0 \leq s \leq \frac{q+1}{2} - t$, $0 \leq r \leq \frac{q+1}{2} - t - s$ and $0 \leq n \leq \frac{q+1}{2} - t - s - r$ and where m_1, m_2, m_3 and m_4 are the powers of z which α, β, γ and δ contribute.

When $n = r = t = 0$ then the monomials occuring are just those from the expansion of $y^q(y^5 + yz^{i+j+k})^{\frac{q+1}{2}}$ and $z^q(y^5 + yz^{i+j+k})^{\frac{q+1}{2}}$. By case G.II) we know that $i+j+k$ must be 5. Suppose that α, β, γ or δ contributes a monomial of the form

$$y^{\frac{7q+5}{2}-4s} z^{5s} \text{ or } y^{\frac{5(q+1)}{2}-4s} z^{q+5s} \notin (y^{2q}, z^{2q}).$$

Then there will exist some pair (u, v, w, x) such that

$$y^{\frac{7q+5}{2}-4s} z^{5s} = y^{\frac{7q+5}{2}-2u-3v-4w-5x} z^{(3+m_1)u+(4+m_2)v+(5+m_3)w+(5+l+m_4)x}$$

or

$$y^{\frac{5(q+1)}{2}-4s} z^{q+5s} = y^{\frac{5(q+1)}{2}-2u-3v-4w-5x} z^{q+(3+m_1)u+(4+m_2)v+(5+m_3)w+(5+l+m_4)x}.$$

Both equalities yield the following system of equations:

$-4s = -2u - 3v - 4w - 5w$ and $5s = (3+m_1)u + (4+m_2)v + 5w + (5+l+m_3)x$.

For $l \geq 2$ $(4(5+l+m_3) - 25), (4(4+m_2) - 15), (4(3+m_1) - 10) > 0$, thus $u = v = x = 0$ and $w = t$. Thus

$$y^q(y^5 + \alpha y^3 z^i + \beta y^2 z^{i+j} + \gamma y z^{i+j+k} + \delta z^{i+j+k+l})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

and

$$z^q(y^5 + \alpha y^3 z^i + \beta y^2 z^{i+j} + \gamma y z^{i+j+k} + \delta z^{i+j+k+l})^{\frac{q+1}{2}} \notin (y^{2q}, z^{2q})$$

for $l \geq 2$. Hence by Remark 3.24, we have case 23). By Lemma 3.41, $k = 1$ reduces to G.X). □

APPENDIX A

Factorization Lemmas

Any two dimensional complete normal Gorenstein domain R over an algebraically closed field k of characteristic p with test ideal contained in the maximal ideal and $\mathfrak{m}^2 \subseteq (y, z)$ where (y, z) is a minimal reduction of the maximal ideal is isomorphic to $k[[x, y, z]]/(x^2 - a)$ where $a \in k[[y, z]]$. To analyze which a force R to have test ideal equal to the maximal ideal we would like to find the nicest expression of a after a change of variables over $k[[y, z]]$. For example, take

$$a = a \textit{ quadratic} + \textit{higher terms}.$$

Since the base field is algebraically closed, we can either factor the quadratic as a product of two distinct linear terms or as a product of two like linear terms, i.e. after some change of variables as yz or y^2. Thus in the case where the leading term is quadratic we want to find the friendliest expression in which to represent $a = f + b$ where f is either yz or y^2 and b is of degree higher than 2. Similarly we want to do the same for a with leading terms which are either a cubic, quartic or quintic. In otherwords we want to find a friendly expression for a when a is among the following lists:

CUBICS	QUARTICS	QUINTICS
$yz(y+z)+b$	$yz(y+z)(y+\lambda z)+b$	$yz(y+z)(y+\lambda z)(y+\mu z)+b$
y^2z+b	$y^2z(y+z)+b$	$y^2z(y+z)(y+\lambda z)+b$
y^3+b	y^2z^2+b	$y^2z^2(y+z)+b$
	y^3z+b	$y^3z(y+z)+b$
	y^4+b	y^3z^2+b
		y^4z+b
		y^5+b

The following lemmas inspired by Exercise I.5.14 in [12] will do just that.

Lemma A.1 *Let $R = k[[y,z]]$. We can rewrite $yz+b$ where $b \in (y,z)^i$ where $i > 2$ as $(y+c)(z+a)$ where $a, c \in (y,z)^{i-1}$. In otherwords after a change of variables, we see $yz+b = y'z' \in k[[y',z']] = k[[y,z]]$.*

Proof. Rewrite

$$yz+b = yz + a_{i-1}y + c_{i-1}z$$

where $a_{i-1}, b_{i-1} \in (y,z)^{i-1}$. Adding zero in the form $a_{i-1}c_{i-1} - a_{i-1}c_{i-1}$ we see that

$$yz+b = (y+c_{i-1})(z+a_{i-1}) - a_{i-1}c_{i-1}.$$

Relabel $(y+c_{i-1})$ as y_1, $(z+a_{i-1})$ as z_1 and $-a_{i-1}c_{i-1}$ as b_1 where $b_1 = a_{i-1}c_{i-1} \in (y,z)^{i+1}$. We see that

$$yz+b = y_1z_1 + b_1.$$

Suppose by induction we can write

$$yz + b = y_r z_r + b_r$$

where $b_r \in (y, z)^{r+i}$. Rewrite $b_r = a_{r+i-1}y + c_{r+i-1}z$. Again adding zero in the form $a_{r+i-1}c_{r+i-1} - a_{r+i-1}c_{r+i-1}$ we see that

$$yz + b = (y_r + c_{r+i-1})(z_r + a_{r+i-1}) - a_{r+i-1}c_{r+i-1} = y_{r+1}z_{r+1} + b_{r+1}$$

where $y_{r+1} = y_r + c_{r+i-1}$, $z_{r+1} = z_r + a_{r+i-1}$ and $b_{r+1} = -a_{r+i-1}c_{r+i-1}$. Since R is complete, $\bigcap(y, z)^r = (0)$. Note $y_r z_r - y_{r+1}z_{r+1} \in \mathfrak{m}^{r+i}$ for all n. Thus $y_r z_r$ converges to $(y + c)(z + a)$ where $a = \Sigma_{\{r \geq i-1\}} a_r$ and $c = \Sigma_{\{r \geq i-1\}} c_r$. Thus

$$yz + b = (y + c)(z + a).$$

□

Lemma A.2 *Let $R = k[[y, z]]$ with char$(R) > 2$. We can rewrite $y^2 + b$ where $b \in (y, z)^i \backslash (y, z)^{i+1}$ where $i > 2$ as $(y + a)^2 + cz^i$ where c is a unit or zero and $a \in (y, z)^{i-1}$. In otherwords, after a change of variables, we see $y^2 + b = y_1^2 + cz^i \in k[[y_1, z]] = k[[y, z]]$.*

Proof. Rewrite $b = a_{i-1}y + c_{01}z^i$ where c_{01} is a unit, zero or is divisible by z. Adding zero in the form $\frac{a_{i-1}}{2}^2 - \frac{a_{i-1}}{2}^2$ we see that

$$y^2 + b = (y + \frac{a_{i-1}}{2})^2 - \frac{a_{i-1}}{2}^2 + c_{01}z^i.$$

Relabel $(y + \frac{a_{i-1}}{2})$ as y_1 and $-\frac{a_{i-1}}{2}^2$ as b_1. Note $b_1 = -\frac{a_{i-1}}{2}^2 \in (y, z)^{i+1}$. Thus

$$y^2 + b = y_1^2 + b_1 + c_{01}z^i.$$

Suppose by induction that

$$y^2 + b = y_r^2 + b_r + c_{0r} z^i$$

where $b_r \in (y,z)^{r+i}$. Rewrite $b_r = a_{r+i-1}y + c_r z^{r+i}$ where c_r is a unit, zero or divisible by z. Adding zero in the form $\frac{a_{n+i-1}}{2}^2 - \frac{a_{n+i-1}}{2}^2$ we see that

$$y^2 + b = (y + \frac{a_{r+i-1}}{2})^2 - \frac{a_{r+i-1}}{2}^2 + (c_{0r} + c_r z^r) z^i.$$

Relabel $(y + \frac{a_{n+i-1}}{2})$ as y_{r+1}, $-\frac{a_{n+i-1}}{2}^2$ as b_{r+1} and $c_{0r} + c_r z^r$ as c_{0r+1} and noting that $b_1 = -\frac{a_{n+i-1}}{2}^2 \in (y,z)^{r+i+1}$. Thus

$$y^2 + b = y_{r+1}^2 + b_{r+1} + c_{0r+1} z^i.$$

Since R is complete, $\bigcap(y,z)^r = (0)$. Since $y_r^2 + c_{0r} z^i - y_{r+1}^2 + c_{0r+1} z^i \in (y,z)^{r+i}$ for all n then $y_r^2 + c_{0r} z^i$ converge to $(y+a)^2 + cz^i$ where $a = \Sigma_{\{r \geq i-1\}} \frac{a_n}{2}^2$ and $c = \Sigma_{\{r \geq i-1\}} c_r$. Thus

$$y^2 + b = (y+a)^2 + cz^i.$$

□

Lemma A.3 *Let $R = k[[y,z]]$. We can rewrite $yz(y+z) + b$ where $b \in (y,z)^4$ as $\mu(y+a)[\alpha(z+c)][(y+a) + \alpha(z+c)]$ where $a, c \in (y,z)^2$ and α and μ are units. Thus after a change of variables, we can rewrite $yz(y+z) + b$ as $\mu y_1 z_1 (y_1 + z_1) \in k[[y_1, z_1]] = k[[y,z]]$.*

Proof. First set $u = z(y+z)$. Note that any element in $(y,z)^4$ can be written $c_3 y + a_2 u$ where $c_3 \in (y,z)^3$ and $a_2 \in (y,z)^2$. Now by the same methods of Lemma

A.1 we can show that $yu + b = (y + a)(u + b')$ where $a \in (y,z)^2$ and $b' \in (y,z)^3$. Applying Lemma A.1, we see that $u + b' = z(y+z) + b' = (z+c)(y+z+d)$. Thus

$$yz(y+z) + b = (y+a)(z+c)(y+z+d).$$

But $d - a - c$ can be written as a $k[[y,z]]$-linear combination of $y + a$ and $z + c$; $d - a - c = e_1(y+a) + e_2(z+c)$. Setting $\beta_1 = 1 + e_1$ and $\beta_2 = 1 + e_2$ which are both units in $k[[y,z]]$ we see that

$$yz(y+z) + b = (y+a)(z+c)[\beta_1(y+a) + \beta_2(z+c)].$$

Setting $\alpha = \beta_2\beta_1^{-1}$ and $\mu = \beta_1^2\beta_2^{-1}$ we see that

$$yz(y+z) + b = \mu(y+a)[\alpha(z+c)][(y+a) + \alpha(z+c)].$$

□

Lemma A.4 *Let $R = k[[y,z]]$ with char$(R) > 2$. We can rewrite $y^2z + b$ where $b \in (y,z)^i \backslash (y,z)^{i+1}$ where $i > 3$ as $(y+a)^2(z+c) + d(z+c)^i$ where d is a unit and $a, c \in (y,z)^{i-2}$. In otherwords, after a change of variables we can replace $y^2z + b$ with $y_1^2 z_1 + dz_1^i \in k[[y_1, z_1]] = k[[y,z]]$.*

Proof. By Lemma A.1 techniques we can rewrite

$$y^2z + b = (y^2 + b')(z + c)$$

where $b' \in (y,z)^{i-1}$ and $c \in (y,z)^{i-2}$. Since y and $z + c$ span (y,z) by Lemma A.2 we can rewrite $(y^2 + b') = (y+a)^2 + d(z+c)^i$ where d is a unit, zero or divisible by $z+c$. Thus

$$y^2z+b=(y+a)^2(z+c)+d(z+c)^i.$$

□

Lemma A.5 *Let $R=k[[y,z]]$ with $char(R)>3$. We can rewrite y^3+b where $b\in(y,z)^i$ where $i>3$ as $(y+a)^3+cz^{i-1}$ where $a\in(y,z)^{i-1}$ and $c\in(y,z)$. In otherwords, after a change of variables we can replace y^3+b with $y_1^3+cz^{i-1}\in k[[y_1,z]]=k[[y,z]]$.*

Proof. Note that any element in $(y,z)^i$ can be written as a linear combination of y^2 and z^{i-1}. Thus if we rewrite $b=a_{i-2}y^2+c_{11}z^{i-1}$ where $a_{i-2}\in(y,z)^{i-2}$ and $c_{11}\in(y,z)$ we see that

$$y^3+b=(y+\frac{a_{i-2}}{3})^3-\frac{(a_{i-2})^2}{3}y-(\frac{a_{i-2}}{3})^3+c_{11}z^{i-1}.$$

Relabel above as $y_1^3+b_1+c_{11}z^{i-1}$. Suppose by induction that $y^3+b=y_r^3+b_r+c_{1r}z^{i-1}$. Then rewrite $b=a_{r+i-2}y_r^2+c_rz^{r+i-1}$ where $a_{r+i-2}\in(y,z)^{r+i-2}$ and $c_r\in(y,z)$ we see that

$$y^3+b=(y_r+\frac{a_{r+i-2}}{3})^3-\frac{(a_{r+i-2})^2}{3}y_r-(\frac{a_{r+i-2}}{3})^3+(c_{1r}+c_rz^r)z^{i-1}.$$

Relabel above as $y_{r+1}^3+b_{r+1}+c_{1r+1}z^{i-1}$. Again since R is complete $y_r^3+c_1nz^{i-1}$ converge to $(y+a)^3+cz^{i-1}$ where $a=\Sigma_{\{r\geq i-1\}}\frac{a_n}{3}$ and $c=\Sigma_{\{r\geq i-1\}}c_{1r}$. Thus

$$y^3+b=(y+a)^3+cz^{i-1}.$$

□

Lemma A.6 *Let $R = k[[y,z]]$. After a change of variables, we can rewrite $yz(y+z)(y+\lambda z)+b$ where $b \in (y,z)^5$ as $\alpha y_1 z_1(y_1+z_1)(y_1+\mu z_1)$ where α and μ are both units.*

Proof. Note that any element of $(y,z)^5$ can be written as a linear combination of y and $z(y+z)(y+\lambda z)$. Thus by techniques similar to those used in proving Lemma A.1, we see that we can rewrite

$$yz(y+z)(y+\lambda z)+b = (y+a)(z(y+z)(y+\lambda z)+b')$$

where $a \in (y,z)^2$ and $c \in (y,z)^4$. From the proof of Lemma A.3, we see that $(z(y+z)(y+\lambda z)+b')$ can be rewritten $(z+c)[(y+z)+d][(y+\lambda z)+e)]$. Thus

$$(yz(y+z)(y+\lambda z)+b = (y+a)(z+c)[(y+z)+d][(y+\lambda z)+e)]$$

$$= (y+a)(z+c)[(y+a)+(z+c)+d-a-c][(y+a)+\lambda(z+c)+e-a-\lambda c)].$$

Again since $d-a-c$ and $e-a-\lambda c$ are both a linear combination of $(y+a)$ and $(z+c)$ then $d-a-c = d_1(y+a)+d_2(z+c)$ and $e-a-\lambda c = e_1(y+a)+e_2(z+c)$. Setting $\delta_i = 1+d_i$ for $i=1,2$, $\epsilon_1 = 1+e_1$ and $\epsilon_2 = \lambda + e_2$, all of which are units we see that

$$yz(y+z)(y+\lambda z)+b = (y+a)(z+c)(\delta_1(y+a)+\delta_2(z+c))(\epsilon_1(y+a)+\epsilon_2(z+c)).$$

Setting $y_1 = y+a, z_1 = \delta_2\delta_1^{-1}(z+c), \alpha = \delta_1^2\epsilon_1\delta_2^{-1}, \mu = \epsilon_2\epsilon_1^{-1}\delta_1\delta_2^{-1}$ then

$$yz(y+z)(y+\lambda z)+b = \alpha y_1 z_1(y_1+z_1)(y_1+\mu z_1).$$

□

Lemma A.7 *Let $R = k[[y,z]]$ with char$(R) > 2$. After a change of variables, we can rewrite $y^2z(y+z)+b$ where $b \in (y,z)^i \backslash (y,z)^{i+1}$ where $i > 4$ as $\alpha(y_1^2 + cz_1^{i-2})z_1(y_1+z_1)$ where α is a unit and c is either a unit or zero.*

Proof. Note that any element of $(y,z)^i$ can be represented as a $k[[y,z]]$-linear equation of y^2 and $z(y+z)$. By techniques similar to those in Lemma A.1 we can rewrite

$$y^2z(y+z)+b = (y^2+b_1)(z(y+z)+b_2)$$

where $b_1, b_2 \in (y,z)^{i-2}$. By Lemma A.1 we can rewrite

$$(z(y+z)+b_2) = (z+a_1)(y+z+a_2)$$

where $a_1, a_2 \in (y,z)^{i-3}$. Since $y, z+a_1$ span (y,z), by Lemma A.2 we can rewrite

$$(y^2+b_1) = (y+d)^2 + c(z+a_1)^{i-2}$$

where $d \in (y,z)^{i-3}$ and c is a unit or zero. Thus

$$y^2z(y+z)+b = [(y+d)^2 + c(z+a_1)^{i-2}](z+a_1)(y+z+a_2).$$

But $a_2 - a_1 - d$ is a $k[[y,z]]$-linear combination of $y+d$ and $z+a_1$; in other words $a_2 - a_1 - d = e_1(y+d) + e_2(z+a_1)$. Setting $\epsilon_i = 1 + e_i$ for $i = 1,2$ we see that

$$y^2z(y+z)+b = [(y+d)^2 + c(z+a_1)^{i-2}](z+a_1)(\epsilon_1(y+d) + \epsilon_2(z+a_1)).$$

Now setting $y_1 = (y+d), z_1 = \epsilon_2\epsilon_1^{-1}(z+a_1), \alpha = \epsilon_1^2\epsilon_2^{-1}$ we see that

$$y^2z(y+z)+b = \alpha(y_1^2 + cz_1^{i-2})z_1(y_1+z_1).$$

□

Lemma A.8 *Let $R = k[[y,z]]$ with char$(R) > 2$. After a change of variables, we can rewrite $y^2z^2 + b$ where $b \in (y,z)^i \backslash (y,z)^{i+1}$ where $i > 4$ as $\alpha(y_1^2 + cz_1^{i-2})(z_1^2 + dy_1^{i-2})$ where α is a unit and c and d are either units or zero.*

Proof. Note that any element of $(y,z)^i$ can be represented as a $k[[y,z]]$-linear equation of y^2 and z^2. By techniques similar to those in Lemma A.1 we can rewrite

$$y^2z^2 + b = (y^2 + b_1)(z^2 + b_2)$$

where $b_1, b_2 \in (y,z)^{i-2}$. By Lemma A.2 we can rewrite

$$(y^2 + b_1) = (y + a_1)^2 + cz^{i-2}$$

where $a_1 \in (y,z)^{i-3}$ and c is a unit or zero. Since $y + a_1, z$ span (y,z), by Lemma A.2 we can rewrite

$$(z^2 + b_2) = (z + a_2)^2 + d(y + a_1)^{i-2}$$

where $a_2 \in (y,z)^{i-3}$ and d is a unit or zero. Thus

$$y^2z^2 + b = [(y + a_1)^2 + cz^{i-2}][(z + a_2)^2 + d(y + a_1)^{i-2}]$$

$$= [(y + a_1)^2 + c(z + a_2)^{i-2} - e][(z + a_2)^2 + d(y + a_1)^{i-2}]$$

where $e = c(\Sigma_{j=0}^{j=i-3}\binom{i-2}{j}z^j a_2^{i-2-j})$. We see that $e \in (y,z)^{2i-6}$ in which every element is a $k[[y,z]]$-linear combination of $(y+a_1)^2$ and $(z+a_2)^{i-2}$; in otherwords, $-e = e_1(y + a_1)^2 + e_2(z + a_2)^{i-2}$. Setting $\beta = 1 + e_1$ and $\gamma = c + e_2$ we see

$$y^2z^2 + b = [\beta(y + a_1)^2 + \gamma(z + a_2)^{i-2}][(z + a_2)^2 + d(y + a_1)^{i-2}].$$

Now setting $y_1 = (y + a_1)$,$z_1 = (z + a_2)$, $\alpha = \beta$ and $c' = \gamma\beta^{-1}$ we get

$$y^2z^2 + b = \alpha(y_1^2 + c'z_1^{i-2})(z_1^2 + dy_1^{i-2}).$$

□

Lemma A.9 *Let $R = k[[y, z]]$ with $char(R) > 3$. After a change of variables, we can rewrite $y^3z + b$ where $b \in (y, z)^i \backslash (y, z)^{i+1}$ where $i > 4$ as $z_1(y_1^3 + cz_1^{i-2})$ where $c \in (y, z)$.*

Proof. Note that any element of $(y, z)^i$ can be represented as a $k[[y, z]]$-linear equation of y^3 and z. By techniques similar to those in Lemma A.1 we can rewrite

$$y^3z + b = (y^3 + b_1)(z + b_2)$$

where $b_1 \in (y, z)^{i-1}$ and $b_2 \in (y, z)^{i-3}$. Since $y, z + b_2$ span (y, z) by Lemma A.5 we can rewrite $y^3 + b_1 = (y + a)^3 + c(z + b_2)^{i-2}$ where c is a unit. Setting $y_1 = y + a$ and $z_1 = z + b_2$ we see that

$$y^3z + b = z_1(y_1^3 + cz_1^{i-2}).$$

□

Lemma A.10 *Let $R = k[[y, z]]$ with $char(R) > i - 2$. After a change of variables, we can rewrite $y^4 + b$ where $b \in (y, z)^i \backslash (y, z)^{i+1}$ where $i > 4$ as $(y_1^2 + cz_1^{i-2})^2 + dz_1^{i-1}$ where c is 1 or 0 and $d \in (y, z)$.*

Proof. Note that any element of $(y,z)^i$ can be represented as a $k[[y,z]]$-linear equation of y^2 and z^{i-1}. By techniques similar to those in Lemma A.2 we can rewrite

$$y^4 + b = (y^2 + a)^2 + dz^{i-1}$$

where $d \in (y,z)$ and $a \in (y,z)^{i-2}$. By Lemma A.2 we can rewrite

$$y^2 + a = (y + a')^2 + cz^{i-1}$$

where c is a unit and $a' \in (y,z)^{i-3}$. Setting $y_1 = y + a'$ and

$$z_1 = \begin{cases} z & \text{if } c = 0 \\ c^{\frac{1}{i-2}} z & \text{if } c \text{ is a unit} \end{cases}$$

we see that

$$y^4 + b = (y_1^2 + cz_1^{i-2})^2 + dz_1^{i-1}$$

where c is either zero or one. □

Lemma A.11 *Let $R = k[[y,z]]$. After a change of variables, we can rewrite $yz(y+z)(y+\lambda z)(y+\mu z)+b$ where $b \in (y,z)^6$ as $\alpha y_1 z_1 (y_1+z_1)(y_1+\lambda_1 z_1)(y_2+\mu_1 z_1)$ where α, λ_1, μ_1 are all units.*

Proof. Note that any element of $(y,z)^6$ can be represented as a $k[[y,z]]$-linear equation of y and $z(y+z)(y+\lambda z)(y+\mu z)$. Thus by techniques similar to Lemma A.1 we can rewrite

$$yz(y+z)(y+\lambda z)(y+\mu z) + b = (y + b_1)(z(y+z)(y+\lambda z)(y+\mu z) + b_2)$$

where $b_1 \in (y,z)^5$ and $b_2 \in (y,z)^2$. From the proof of Lemma A.6 we can rewrite

$$z(y+z)(y+\lambda z)(y+\mu z)+b_2 = (z+a_1)(y+z+a_2)(y+\lambda z+a_3)(y+\mu z+a_4).$$

Since $y+b_1$ and $z+a_1$ span (y,z) we can rewrite $a_2-a_1-b_1, a3-\lambda a_1-b_1, a_4-\mu a_1-b_1$ as a $k[[y,z]]$-linear combination of $y+b_1$ and $z+a_1$. In other words $a_2-a_1-b_1 = c_1(y+b_1)+c_2(z+a_1)$, $a_3-\lambda a_1-b_1 = d_1(y+b_1)+d_2(z+a_1)$ and $a_4-\mu a_1-b_1 = e_1(y+b_1)+e_2(z+a_1)$. Setting $1+c_1=\gamma_1$, $1+c_2=\gamma_2$, $1+d_1=\delta_1$, $\lambda+d_2=\delta_2$, $1+e_1=\epsilon_1$, and $\mu+e_2=\epsilon_2$ we see that

$yz(y+z)(y+\lambda z)(y+\mu z)+b=$

$(y+b_1)(z+a_1)(\gamma_1(y+b_1)+\gamma_2(z+a_1))(\delta_1(y+b_1)+\delta_2(z+a_1))(\epsilon_1(y+b_1)+\epsilon_2(z+a_1)).$

Now setting $y_1=(y+b_1)$, $z_1=\gamma_2\gamma_1^{-1}(z+a_1)$, $\alpha=\gamma_1^2\gamma_2^{-1}\delta_1\epsilon_1$, $\lambda_1=\delta_2\delta_1^{-1}$ and $\mu_1=\epsilon_2\epsilon_1^{-1}$ we see that

$$yz(y+z)(y+\lambda z)(y+\mu z)+b = \alpha y_1 z_1(y_1+z_1)(y_1+\lambda_1 z_1)(y_2+\mu_1 z_1).$$

□

Lemma A.12 *Let $R=k[[y,z]]$ with char$(R)>2$. After a change of variables, we can rewrite $y^2z(y+z)(y+\lambda z)+b$ where $b\in(y,z)^i\backslash(y,z)^{i+1}$ where $i>5$ as $\alpha(y_1^2+cz_1^{i-3})z_1(y_1+z_1)(y_1+\lambda_1 z_1)$ where α,λ_1 are units and c is either zero or a unit.*

Proof. Note that any element of $(y,z)^i$ can be represented as a $k[[y,z]]$-linear equation of y^2 and $z(y+z)(y+\lambda z)$. Thus by techniques similar to Lemma A.1 we can rewrite

$$y^2z(y+z)(y+\lambda z)+b=(y^2+b_1)(z(y+z)(y+\lambda z)+b_2)$$

where $b_1 \in (y,z)^{i-3}$ and $b_2 \in (y,z)^{i-2}$. From the proof of Lemma A.3, we see that

$$(z(y+z)(y+\lambda z)+b_2)=(z+c)[(y+z)+d][(y+\lambda z)+e)].$$

By Lemma A.2 we can rewrite

$$y^2+b_1=(y+a)^2+c'(z+c)^{i-3}$$

where c' is a unit or zero. Thus

$$y^2z(y+z)(y+\lambda z)+b=((y+a)^2+c'(z+c)^{i-3})(z+c)[(y+z)+d][(y+\lambda z)+e)]$$

$$=((y+a)^2+c'(z+c)^{i-3})(z+c)[(y+a)+(z+c)+d-a-c][(y+a)+\lambda(z+c)+e-a-\lambda c)].$$

Again since $d-a-c$ and $e-a-\lambda c$ are both a linear combination of $(y+a)$ and $(z+c)$ then $d-a-c=d_1(y+a)+d_2(z+c)$ and $e-a-\lambda c=e_1(y+a)+e_2(z+c)$. Setting $\delta_i=1+d_i$ for $i=1,2$ and $\epsilon_1=1+e_1$ and $\epsilon_2=\lambda+e_2$, all of which are units we see that

$$y^2z(y+z)(y+\lambda z)+b=$$

$$((y+a)^2+c'(z+c)^{i-3})(z+c)(\delta_1(y+a)+\delta_2(z+c))(\epsilon_1(y+a)+\epsilon_2(z+c)).$$

Setting $y_1=y+a, z_1=\delta_2\delta_1^{-1}(z+c), \alpha=\delta_1^2\epsilon_1\delta_2^{-1}, \lambda_1=\epsilon_2\epsilon_1^{-1}\delta_1\delta_2^{-1}$ then

$$y^2z(y+z)(y+\lambda z)+b=\alpha(y_1^2+c'z_1^{i-3})z_1(y_1+z_1)(y_1+\lambda_1 z_1).$$

□

Lemma A.13 *Let $R = k[[y, z]]$ with $char(R) > 2$. After a change of variables, we can rewrite $y^2z^2(y + z) + b$ where $b \in (y, z)^i \backslash (y, z)^{i+1}$ where $i > 5$ as $\alpha(y_1^2 + cz_1^{i-3})(z_1^2 + dz^{i-3})(y_1 + z_1)$ where α is a unit and c and d are units or zero.*

Proof. Note that any element of $(y, z)^i$ can be represented as a $k[[y, z]]$-linear equation of y^2 and $z^2(y + z)$. Thus by techniques similar to Lemma A.1 we can rewrite

$$y^2z^2(y + z) + b = (y^2 + b_1)(z^2(y + z) + b_2)$$

where $b_1 \in (y, z)^{i-3}$ and $b_2 \in (y, z)^{i-2}$. As in the proof of Lemma A.4 we can rewrite

$$z^2(y + z) + b_2 = (z^2 + a_1)(y + z + a_2)$$

where $a_1 \in (y, z)^{i-3}$ and $a_2 \in (y, z)^{i-4}$. Thus

$$y^2z^2(y + z) + b = (y^2 + b_1)(z^2 + a_1)(y + z + a_2).$$

As in the proof of Lemma A.8 there exist a change of variables y_1, z_1 and unit α and with c and d either units or zero such that

$$(y^2 + b_1)(z^2 + a_1) = \alpha(y_1^2 + cz_1^{i-2})(z_1^2 + dy_1^{i-2}).$$

Under this change of variables we can rewrite

$$(y + z + a_2) = \beta_1 y_1 + \beta_2 z_1$$

with β_i units for $i = 1, 2$. Setting $z_2 = \beta_2\beta_1^{-1}$, $\alpha' = \beta_1^3\alpha\beta_2^{-2}$, $c' = c\beta_1^3\beta_2^{-3}$ and $d' = d\beta_2^2\beta_1^{-2}$ we see that $y^2z^2(y + z) + b = \alpha'(y_1^2 + c'z_2^3)(z_2^2 + d'y_1^3)(y_1 + z_2)$. □

Lemma A.14 *Let $R = k[[y,z]]$ with $char(R) > 3$. After a change of variables, we can rewrite $y^3 z(y+z) + b$ where $b \in (y,z)^i \backslash (y,z)^{i+1}$ where $i > 5$ as $\alpha(y_1^3 + cz_1^{i-3})z_1(y_1 + z_1)$ where α is a unit and $c \in (y,z)$.*

Proof. Note that any element of $(y,z)^i$ can be represented as a $k[[y,z]]$-linear equation of y^3 and $z(y+z)$. Thus by techniques similar to Lemma A.1 we can rewrite

$$y^3 z(y+z) + b = (y^3 + b_1)(z(y+z) + b_2)$$

where $b_1 \in (y,z)^{i-2}$ and $b_2 \in (y,z)^{i-3}$. By Lemma A.1 we can rewrite

$$(z(y+z) + b_2) = (z + a_1)(y + z + a_2)$$

where $a_j \in (y,z)^{i-4}$ for $j = 1,2$. Since $y, z + a_1$ spans (y,z) by Lemma A.5 we can rewrite

$$(y^3 + b_1) = (y + a_3)^3 + c(z + a_1)^{i-3}$$

where $c \in (y,z)$. We can write $a_2 - a_1 - a_3$ as a $k[[y,z]]$-linear combination of $y + a_3, z + a_1$. In otherwords, $a_2 - a_1 - a_3 = d_1(y + a_3) + d_2(z + a_1)$. Setting $\delta_j = 1 + d_j$ for $j = 1,2$ we see that $(y + z + a_2) = \delta_1(y + a_3) + \delta_2(z + a_1)$. Now setting $y_1 = y + a_3$, $z_1 = \delta_2\delta_1^{-1}(z + a_1)$, $c' = c\delta_1^3\delta_2^{-3}$ and $\alpha = \delta_1^2\delta_2^{-1}$ we see that

$$y^3 z(y+z) + b = \alpha(y_1^3 + c' z_1^{i-3})z_1(y_1 + z_1).$$

□

Lemma A.15 *Let $R = k[[y,z]]$ with $char(R) > 3$. After a change of variables, we can rewrite $y^3z^2 + b$ where $b \in (y,z)^i \backslash (y,z)^{i+1}$ where $i > 5$ as $\alpha(y_1^3 + cz_1^{i-3})(z_1^2 + dy_1^{i-3})$ where α a unit, d either zero or a unit and $c \in (y,z)$.*

Proof. Note that any element of $(y,z)^i$ can be represented as a $k[[y,z]]$-linear equation of y^3 and z^2. Thus by techniques similar to Lemma A.1 we can rewrite

$$y^3z^2 + b = (y^3 + b_1)(z^2 + b_2)$$

where $b_1 \in (y,z)^{i-2}$ and $b_2 \in (y,z)^{i-3}$. By Lemma A.2 we can rewrite

$$(z^2 + b_2) = (z + a_1)^2 + d_1 y^{i-3}$$

where $a_1 \in (y,z)^{i-4}$ and d_1 is a unit or zero. Since $y, z + a_1$ spans (y,z) by Lemma A.5 we can rewrite

$$(y^3 + b_1) = (y + a_2)^3 + c_1(z + a_1)^{i-3}$$

where $c_1 \in (y,z)$. As in the proof of Lemma A.8 we can now rewrite

$$(z + a_1)^2 + d_1 y^{i-3} = \beta_1(z + a_1)^2 + \beta_2(y + a_2)^{i-3}$$

for units β_j, $j = 1,2$. Setting $\alpha = \beta_1$, $y_1 = y + a_2$, $z_1 = z + a_1$, $c = c_1$ and $d = \beta_2\beta_1^{-1}$ we see that

$$y^3z^2 + b = \alpha(y_1^3 + cz_1^{i-3})(z_1^2 + dy_1^{i-3}).$$

□

Lemma A.16 *Let $R = k[[y,z]]$ with char$(R) > i-2$. After a change of variables, we can rewrite $y^4z + b$ where $b \in (y,z)^i \backslash (y,z)^{i+1}$ where $i > 5$ as $z_1[(y_1^2 + cz_1^{i-2})^2 + dz_1^{i-1}]$ where c is one or zero and $d \in (y,z)$.*

Proof. Note that any element of $(y,z)^i$ can be represented as a $k[[y,z]]$-linear equation of y^4 and z. Thus by techniques similar to Lemma A.1 we can rewrite

$$y^4z + b = (y^4 + b_1)(z + b_2)$$

where $b_1 \in (y,z)^{i-1}$ and $b_2 \in (y,z)^{i-4}$. Since $y, z + b_2$ span (y,z) by Lemma A.10 we can rewrite

$$y^4 + b_1 = ((y + a_1)^2 + c(z + b_2)^{i-3})^2 + d(z + b_2)^{i-2}$$

where $a_1 \in (y,z)^{i-2}$, c is a one or zero and $d \in (y,z)$. Setting $y_1 = y + a_1$ and $z_1 = z + b_2$ we see that

$$y^4z + b = z_1[(y_1^2 + cz_1^{i-2})^2 + dz_1^{i-1}].$$

□

Lemma A.17 *Let $R = k[[y,z]]$ with char$(R) > 5$. After a change of variables, we can rewrite $y^5 + b$ where $b \in (y,z)^i \backslash (y,z)^{i+1}$ where $i > 5$ as $y^5 + \alpha y^3 z^{i-1} + \beta y^2 z^{i-2} + \gamma y z^{i-1} + \delta z^i$ where at least one of the α, β, γ or δ is a unit.*

Proof. Note that any element of $(y,z)^i$ can be represented as a $k[[y,z]]$-linear equation of y^5, y^4z^{i-4}, y^3z^{i-3}, y^2z^{i-2}, yz^{i-1} and z^i. Thus

$$y^5 + b = a_5y^5 + a_4y^4z^{i-4} + a_3y^3z^{i-3} + a_2y^2z^{i-2} + a_1yz^{i-1} + a_0z^i.$$

Note that a_5 must be a unit. Dividing by a_5 we get

$$y^5 + b = y^5 + b_4 y^4 z^{i-4} + b_3 y^3 z^{i-3} + b_2 y^2 z^{i-2} + b_1 y z^{i-1} + b_0 z^i$$

where $b_j = \frac{a_j}{a_5}$ for $0 \le j \le 4$. Now set $y_1 = y + \frac{b_4}{5} z^{i-4}$ and suppose c_j are the transformed coefficients of $y^j z^{i-j}$. Then

$$y^5 + b = y^5 + c_3 y^3 z^{i-3} + c_2 y^2 z^{i-2} + c_1 y z^{i-1} + c_0 z^i.$$

And one of the c_j must be a unit. □

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