Collapse in a forced three dimensional nonlinear Schrödinger equation

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We derive sufficient conditions for the occurrence of collapse in a forced three dimensional nonlinear Schrödinger equation without dissipation. Numerical studies continue the results to the case of finite dissipation.

Wave collapse or singularity formation in a finite time in the solution of nonlinear partial differential equations describing systems of dispersive waves is a striking and general phenomenon of nonlinear physics [1]. The nonlinear Schrödinger equation (NLS) is a universal model of weakly nonlinear wave evolution, and is known to lead to collapse when the dimensionality of the problem is at least two [2]. Growth of amplitudes of collapsing waves is accompanied with dramatic contraction of the wave packet. In many cases the underlying physical system is dissipative so that it is natural to account for a source of energy. A generalized forced NLS (FNLS) that includes forcing and damping terms can be written in the form

$$i\frac{\partial\psi}{\partial t} = b\psi - \nabla^2\psi - |\psi|^2\psi + E. \qquad (1)$$

Here $b = b_r + ib_i$ is a complex constant, E is real, and ∇^2 operates in $1, 2$ or 3 dimensions. When $b = E = 0$ we regain the canonical NLS. Equation (1) has been used in 1 and 2 dimensions to describe the dynamics of wave packets and solitons in plasmas, fluids, Josephson junctions, and optical problems [3].

The present work has been motivated by recent studies of three dimensional space-time focusing and structure formation in nonlinear optical cavities pumped by an external train of pulses [5,6]. In that context collapse is an effective mechanism for generating ultrashort pulses from initially smooth wave packets. ψ represents the slowly varying amplitude of the electric field, the medium is assumed to exhibit anomalous dispersion, and the three dimensional Laplacian operates on $\mathbf{r} = (x_1, x_2, x_3)$, where x_1, x_2 are two transverse spatial coordinates and x_3 is a slow temporal coordinate that describes the longitudinal extent of the pulse in a frame traveling with the group velocity. The real part of b is proportional to the phase shift suffered by the field in one cavity round trip. The imaginary part of b is positive if the optical cavity includes an energy source that amplifies the circulating pulse. In the case of a passive cavity with losses due to absorption and/or transmission of the beam through the cavity mirrors $b_i < 0$. Finally E is proportional to the amplitude of the external beam driving the cavity.

Collapse dynamics may be considerably different in the NLS and FNLS. In two dimensions the NLS is critical and localized solutions are at best marginally stable [4]; in the presence of perturbations they either decay or collapse in a finite time. However, in the two-dimensional FNLS numerical results support the possible existence of stable localized solutions [7]. In three dimensions the NLS is supercritical and there are no stable localized solutions [4]. Collapse dynamics in the three-dimensional FNLS have not been investigated previously. Here we prove analytically in 3 dimensions that collapse takes place in the FNLS with zero dissipation $b \equiv b_r$ under some integral restrictions on the initial conditions. Numerical studies confirm that collapse can also occur for both signs of b_i.

For real b the steady state plane wave solution ψ_0 of (1) is real and governed by the equation

$$\psi_0^3 - b\psi_0 - E = 0. \qquad (2)$$

For $b \le 3(\frac{E}{2})^{\frac{2}{3}}$ there is only one solution of (2). This solution is linearly stable with respect to space-homogeneous perturbations. Nevertheless for inhomogeneous perturbations $\delta\psi \propto e^{i\mathbf{kr}}$ there always exists a nonzero wave vector $\mathbf{k}$ for which (2) is unstable. For $b > 3(\frac{E}{2})^{\frac{2}{3}}$ there are three solutions of (2). Two of them are linearly stable with respect to space-homogeneous perturbations and the third one is unstable. Among the two stable solutions one is unstable with respect to perturbations with nonzero $\mathbf{k}$, but the second solution $\psi_0 = 2\sqrt{\frac{b}{3}}\cos\frac{\phi-2\pi}{3}$, $\phi = \arctan\sqrt{\frac{4b^3}{27E^2} - 1}$ with asymptotic $\psi_0 \to -1/b$ for $b \to \infty$ is stable for all values of $\mathbf{k}$.

We consider the FNLS in a finite box of size L : $(-\frac{L}{2} \le x_j \le \frac{L}{2},\quad j = 1, 2, 3)$. with the boundary conditions on the surface of the box corresponding to steady state solution (2):

$$\psi\Big|_{surface} = \psi_0, \quad \frac{\partial\psi}{\partial x_j}\Big|_{surface} = 0, \quad j = 1, 2, 3 \qquad (3)$$

The FNLS can be written in the Hamiltonian form $i\psi_t = \frac{\delta H}{\delta\psi^*}$, where Hamiltonian

$$H = \int \left[|\nabla\psi|^2 - \frac{|\psi|^4}{2} + b|\psi|^2 + E(\psi + \psi^*) \right] d^3x \qquad (4)$$

is an integral of motion but the number of particles $N = \int |\psi|^2 d^3x$ is no longer an integral of the motion since $N_t = iE\int(\psi-\psi^*)d^3x$ contrary to the usual NLS where $N_t = 0$.

To find a sufficient collapse condition in the FNLS consider the temporal evolution of the quantity $A = \int r^2|\psi|^2 d^3x, \quad r^2 = x_j^2$ (repeated index j means summation over all space coordinates $j = 1\dots 3$). A/N is the average width of the distribution of ψ or simply $< r^2 >$ in the quantum mechanical interpretation of the FNLS. Using (1), integrating by parts, and taking into account boundary conditions (3) we get for the first time derivative

$$A_t = \int \left[2ix_j(\psi\partial_j\psi^* - \psi^*\partial_j\psi) + ir^2E(\psi-\psi^*)\right]d^3x. \quad (5)$$

In a similar way after a second differentiation by t we get

$$A_{tt} = 2\alpha H - (2\alpha-8)\int|\nabla\psi|^2d^3x - (6-\alpha)\int|\psi|^4d^3x - 2\alpha b\int|\psi|^2d^3x - E(12+2\alpha)\int(\psi+\psi^*)d^3x$$
$$-E\int r^2\Big[|\psi|^2(\psi+\psi^*) - b(\psi+\psi^*) - 2E\Big]d^3x + 6L^3\psi_0(\psi_0^3+4E), \quad (6)$$

where the sum of all terms proportional to α is identically zero (see (4)). Thus α is an arbitrary real number. We will be interested in the range $4 \le \alpha \le 6$ where both the second and third terms on the right-hand side of (6) are not positive, and they can be effectively used for estimating a bound on A_{tt} from above.

Following analysis of the usual NLS we refer to this expression as the virial theorem. Note that in the case of $E = 0$ we return to the virial theorem for the NLS [8,4]. Below we suppose that $E \ne 0$ which allows us by proper rescaling of b, ψ, $\mathbf{r}$, and t to set $E = 1$ without loss of generality. In order to establish a sufficient condition for collapse we bound (6) from above and find an integral estimate for initial conditions of the FNLS for which A becomes negative in a finite time. Because A is a positive-definite function this means singularity formation in the solution of the FNLS together with catastrophic squeezing of the distribution of $|\psi|$.

Bounding (6) from above we get

$$A_{tt} \le 2\alpha H - (2\alpha-8)\int|\nabla\psi|^2d^3x - (6-\alpha)\int|\psi|^4d^3x$$
$$-2\alpha b\int|\psi|^2d^3x + 4(6+\alpha)\int|\psi|d^3x$$
$$+\int r^2\Big[2|\psi|^3 + 2|b||\psi| + 2\Big]d^3x + 6L^3\psi_0(\psi_0^3+4). \quad (7)$$

In contrast to the NLS we can prove a sufficient collapse condition for the FNLS only in a finite box. It is possible to bound positive-definite terms on the right-hand side of (7) by different approaches. Our primary aim below is to get a sufficient collapse condition for the largest possible values of L. We use a number of inequalities that follow from the Cauchy-Schwarz inequality in a finite box:

$$\int r^2|\psi|^3d^3x \le \frac{\sqrt{3}L}{2}\int r|\psi|^3d^3x \le \frac{\sqrt{3}L}{2}A^{\frac12}\Big(\int|\psi|^4d^3x\Big)^{\frac12}, \quad (8a)$$

$$\int|\psi|d^3x \le \Big(\int\frac{1}{r^2}d^3x\Big)^{\frac12}A^{\frac12} \le \big(2\pi\sqrt{3}L\big)^{\frac12}A^{\frac12}, \quad (8b)$$

$$\int r^2|\psi|d^3x \le \Big(\int r^2d^3x\Big)^{\frac12}A^{\frac12} \le \frac{L^{\frac52}}{2}A^{\frac12}. \quad (8c)$$

We need additionally to estimate N which can be done by integration by parts and applying the Cauchy-Schwarz inequality [9]

$$N = -\frac23\int x_j\frac{\partial|\psi|^2}{\partial x_j}d^3x \le \frac23A^{\frac12}\Big(\int\Big|\nabla|\psi|\Big|^2d^3x\Big)^{\frac12}$$
$$\le \frac23A^{\frac12}\Big(\int\Big|\nabla\psi\Big|^2d^3x\Big)^{\frac12}. \quad (9)$$

Using $(8a)-(8c)$ and introducing the notation $X = \int|\nabla\psi|^2d^3x, \quad Y = \int|\psi|^4d^3x$ we get

$$A_{tt} \le 2\alpha H - (2\alpha-8)X - (6-\alpha)Y - 2\alpha bN$$
$$+4(6+\alpha)(2\pi\sqrt{3}L)^{\frac12}A^{\frac12} + \sqrt{3}LA^{\frac12}Y^{\frac12}$$
$$+|b|L^{\frac52}A^{\frac12} + \frac{L^5}{2} + 6L^3\psi_0(\psi_0^3+4). \quad (10)$$

But

$$-pY + qY^{\frac12} = -p(Y^{\frac12} - \frac{q}{2p})^2 + \frac{q^2}{4p} \le \frac{q^2}{4p} \quad (11)$$

for arbitrary real Y, q and $p > 0$. Thus we have

$$A_{tt} \le 2\alpha H - (2\alpha-8)X - 2\alpha bN + \frac{3L^2}{4(6-\alpha)}A$$
$$+\big[4(6+\alpha)(2\pi\sqrt{3}L)^{\frac12} + |b|L^{\frac52}\big]A^{\frac12} + \frac{L^5}{2}$$
$$+6L^3\psi_0(\psi_0^3+4). \quad (12)$$

For $b \ge 0$ we set $\alpha = 4$ and bound the right-hand side of this inequality from above using the inequality $A \le 3L^2N/4$

$$A_{tt} \le 8H + \big(-\frac{32}{3L^2}b + \frac{3L^2}{8}\big)A + \big(40(2\pi\sqrt{3}L)^{\frac12} + |b|L^{\frac52}\big)A^{\frac12} + \frac{L^5}{2} + 6L^3\psi_0(\psi_0^3+4). \quad (13)$$

For $b < 0$ we can estimate N in (12) via (9) to get

$$A_{tt} \le 2\alpha H - (2\alpha - 8)X - 2\alpha b\frac{2}{3}A^{\frac{1}{2}}X^{\frac{1}{2}} + \frac{3L^2}{4(6-\alpha)}A + \left[4(6+\alpha)(2\pi\sqrt{3}L)^{\frac{1}{2}} + |b|L^{\frac{5}{2}}\right]A^{\frac{1}{2}} + \frac{L^5}{2} + 6L^3\psi_0(\psi_0^3+4). \quad (14)$$

In turn from (11) (where we use N instead of Y) we obtain

$$A_{tt} \le 2\alpha H + \Big(\frac{2\alpha^2 b^2}{9(\alpha-4)} + \frac{3L^2}{4(6-\alpha)}\Big)A + \left[4(6+\alpha)(2\pi\sqrt{3}L)^{\frac{1}{2}} + |b|L^{\frac{5}{2}}\right]A^{\frac{1}{2}} + \frac{L^5}{2} + 6L^3\psi_0(\psi_0^3+4). \quad (15)$$

To get the best estimate it is necessary to find the minimum of this expression as a function of α on the interval $4 \le \alpha \le 6$. But an analytical expression for the minimum position for arbitrary values of parameters b, L, A is too cumbersome to be written here explicitly. Instead we set below $\alpha = 5$ keeping in mind however that this is not the strictest possible estimate.

Both differential inequalities (13) and (15) (for $\alpha = 5$) can be rewritten as

$$A_{tt} = -\frac{\partial U(A)}{\partial A} - g^2(t), \quad (16)$$

where

$$U(A) = -w_0 A - \frac{w_1}{2}A^2 - \frac{2w_2 A^{\frac{3}{2}}}{3}, \quad (17)$$

$$w_0 = \begin{cases} 8H + \frac{L^5}{2} + 6L^3\psi_0(\psi_0^3+4), & b \ge 0 \\ 10H + \frac{L^5}{2} + 6L^3\psi_0(\psi_0^3+4), & b < 0 \end{cases}$$
$$w_1 = \begin{cases} -\frac{32}{3L^2}b + \frac{3L^2}{8}, & b \ge 0 \\ \frac{50b^2}{9} + \frac{3L^2}{4}, & b < 0 \end{cases} \quad (18)$$
$$w_2 = \begin{cases} 40(2\pi\sqrt{3}L)^{\frac{1}{2}} + |b|L^{\frac{5}{2}}, & b \ge 0 \\ 44(2\pi\sqrt{3}L)^{\frac{1}{2}} + |b|L^{\frac{5}{2}}, & b < 0 \end{cases}$$

and $g^2(t)$ is some unknown nonnegative function of time. Equation (16) has a simple mechanical analogy [9] with the motion of a "particle" with coordinate A under the influence of the potential force $-\frac{\partial U(A)}{\partial A}$ in addition to the force $-g^2(t)$. Due to the influence of the nonpotential force $-g^2(t)$ the total energy $\mathcal{E}$ of the particle is time dependent: $\mathcal{E}(t) = \frac{A_t^2}{2} + U(A)$. Collapse certainly occurs if the "particle" reaches the origin $A = 0$. It is clear that if the particle were to reach the origin without the influence of the force $-g^2(t)$ then it would reach the origin even faster under the additional influence of this nonpositive force. Therefore we consider below the particle dynamics without the influence of the nonconservative force $-g^2(t)$.

It follows from (18) that $w_2 > 0$ for all values of parameters b, L thus we can classify the potential $U(A)$ depending on the signs of $w_0,\ w_1$ and $w_2^2 - 4w_0w_1$ (see fig. 1). In particular for $w_0 < 0,\ w_1 < 0,\ w_2^2 > 4w_0w_1$ (curve 2) or $w_0 < 0,\ w_1 > 0$ (curve 4) the potential has a barrier at

$$A_m = \Big(\frac{\sqrt{w_2^2 - 4w_0w_1} - w_2}{2w_1}\Big)^2 \quad (19)$$

with particle energy $\mathcal{E}_m = U(A_m)$ at the top. In the other cases (curves 1,3,5) there is no barrier. Thus we can separate sufficient collapse conditions into four different cases:

a) for $w_0 \ge 0,\ w_1 \ge 0,\ \mathcal{E}(0) > 0,\ A_t\big|_{t=0} < 0$ the particle reaches the origin in a finite time irrespective of the initial value of $A\big|_{t=0}$;

b) for either $w_0 \ge 0,\ w_1 < 0,\ \mathcal{E}(0) > 0$ or $w_0 < 0,\ w_1 < 0,\ w_2^2 < 4w_0w_1$ the particle reaches the origin in a finite time for all possible initial values of $A\big|_{t=0}$ and $A_t\big|_{t=0}$;

c) for either $w_0 < 0,\ w_1 < 0,\ w_2^2 > 4w_0w_1$ or $w_0 < 0,\ w_1 > 0$ together with conditions $A\big|_{t=0} < A_m,\ \mathcal{E}(0) < \mathcal{E}_m$ the particle can not overcome the barrier from left to right thus it always falls to the origin in a finite time;

d) for either $w_0 < 0,\ w_1 < 0,\ w_2^2 > 4w_0w_1$ or $w_0 < 0,\ w_1 > 0$ together with conditions $\mathcal{E}(0) > \mathcal{E}_m,\ A_t\big|_{t=0} < 0$ the particle is able to overcome the barrier thus it always falls to the origin in a finite time irrespective of the initial value of $A\big|_{t=0}$.

Note that we prove analytically only sufficient collapse conditions. It means that even if none of conditions a,b,c,d are satisfied we can not exclude collapse formation for some particular values of the initial conditions of (1). To find a strict boundary of collapse formation we have assumed radial symmetry and integrated Eq. (1) on the domain $0 \le r < \frac{L}{2}$ with the boundary condition $\psi_r\big|_{r\to\frac{L}{2}} = 0$ and Gaussian-like initial condition $\psi|_{t=0} = \psi_0 + p\left(e^{-\beta^2r^2} + (\beta^2r^2 - 1 - \beta^2\frac{L^2}{4})e^{-\beta^2L^2/4}\right)$, where p and β are arbitrary complex and real parameters respectively. We suppose that $e^{-\beta^2L^2/4} \ll 1$ thus the difference between boundary conditions used in numerics and (3) is exponentially small. We set $b > \frac{3}{2^{\frac{2}{3}}}$ and choose ψ_0 corresponding to the stable branch of (2). For the initial amplitude we use $p = |p|e^{\imath \arg(\psi_0)}$. Figures 2a and 2b give the dependence of collapse threshold amplitude p_{thresh} on β and L obtained numerically and analytically from the sufficient collapse criteria. Note that depending on the parameters the analytical value of the threshold corresponds to different cases a),b),c), or d). The shape of the collapse threshold curves found analytically and numerically is similar, although they differ in amplitude by a numerical factor of order 5.

For $b \le \frac{3}{2^{\frac{2}{3}}}$ our sufficient collapse criterion can also predict collapse but numerical simulations assuming radial symmetry are not useful because any background solution of (2) is modulationally unstable. Thus any general perturbation that breaks the radial symmetry will grow at least exponentially in time. In the NLS the nonlinear stage of the modulational instability results in a set of collapsing filaments and we expect a similar scenario here. Thus we would need to make full 3D simulations of collapse formation which are computationally very expensive, especially near singularity points. Thus in that case our collapse criterion is especially helpful.

Experimental observation of three dimensional solitons in optical cavities will realistically require finite dissipation and $b_i \neq 0$. The effect of dissipation on collapse is shown in Fig. 3 where $|p_{thresh}(b_i)|$ is given for several values of b_r and β. The collapse threshold increases as the dissipation is raised.

From a physical point of view it is clear that collapse can also occur in the limit $L \to \infty$ because for rapidly decaying initial conditions $\psi\big|_{|\mathbf{r}|\to\infty} \to 0$ the tails of $|\psi|$ have no influence on collapse. But we can analytically prove sufficient collapse conditions only for finite L. Nevertheless the sufficient collapse criterion can predict collapse for so large L that all differences between collapse in a finite box and in an infinite domain will be determined by exponentially small tails of the $|\psi|$ distribution. The collapse of course occurs in this case and numerical simulations support that conclusion.

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FIG. 1. Typical behaviour of potential $U(A)$ from Eq. (17) for five qualitatively different cases: $w_0 > 0, w_1 > 0$ (curve 1); $w_0 < 0, w_1 < 0, w_2^2 > 4w_0w_1$ (curve 2); $w_0 < 0, w_1 < 0, w_2^2 < 4w_0w_1$ (curve 3); $w_0 < 0, w_1 > 0$ (curve 4); $w_0 > 0, w_1 < 0$ (curve 5).

FIG. 2. Collapse threshold found numerically (dotted curves with symbols) and analytically (solid curves) as a function of L a) and β b), for $b = 2, 10$.

FIG. 3. Influence of dissipation on the collapse threshold for $b_r = 2$ (circles) and $b_r = 10$ (squares) with $L = 5$. Solid curves $\beta = 1$, dashed curves $\beta = 2$, and dotted curves $\beta = 0.5$.