MATH 561 - COMPLEX ANALYSIS I
Homework # 1:

HE = Hahn-Epstein's book, GK = Greene-Krantz's book.

1. (HE Exercise 3, p. 12) Prove the *parallelogram law*:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2),$$

for two arbitrary complex numbers z, w. Interpret the equality geometrically.

2. (HE Exercise 5, p. 13) If $|\alpha| < 1$ and $|z| \le 1$, show that

$$\left|\frac{z-\alpha}{1-\overline{\alpha}z}\right| \le 1.$$

When does equality hold?

3. (GK Exercise 7, p. 21) Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ have real coefficients $a_j \in \mathbb{R}$, $0 \le j \le n$. Prove that if z_0 is a root pf the polynomial p (i.e. $p(z_0) = 0$), then also $\overline{z_0}$ is a root of p. Give a counterexample when the coefficients are not all real.

4. (GK Exercise 8, p. 21) A field F is said to be *ordered* if there is a distinguished subset $P \subset F$ such that:

(i) if $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;

(ii) if $a \in F$ then precisely one of the following holds: $\quad a \in P \quad$ or $\quad -a \in P \quad$ or $\quad a = 0$.

Verify that $\mathbb{R}$ is an ordered field when P is chosen to be the strictly positive real numbers. Prove that $\mathbf{C}$ is not an ordered field.

5. (GK Exercise 9, p. 21) Show that the function

$$\phi(z) = i\frac{1-z}{1+z}$$

maps the open unit disc $D = \{z : |z| < 1\}$ one-to-one onto the upper half plane $U = \{z = x + iy : y > 0\}$. (The map ϕ is called the *Cayley transform.*)

6. (HE Exercise 14(a), p. 14) Derive the identity

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[(n+\frac{1}{2})\theta\right]}{2\sin\frac{\theta}{2}} = \frac{\cos\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}},$$

for $0 < \theta < 2\pi$.

This homework is due on Monday August 26.