**Reading Instructions for Final (Thursday May 11, 10:00–12:00, in class)**

- Read your lecture notes.
- Read the following sections of the textbook:

If you have the **10th edition**:

- **Ch. 7**: Sec. 7.1
  Sec. 7.2 (except Example 2)
  Sec. 7.3 (except formula (7) and Examples 1, 2, 5)
  Sec. 7.4 (except Theorem 7.4.4)
  Sec. 7.5
  Sec. 7.6
  Sec. 7.8
  Sec. 7.9 (only the method of variation of parameters on pages 443–445)
- **Ch. 8**: *Instead of the textbook, read pages 1–20 of the lecture notes posted online*
- **Ch. 9**: Sec. 9.1
  Sec. 9.2 (except pages 511–514)
  Sec. 9.3 (only page 522 on Jacobian matrix)
  Sec. 9.5 (except pages 549–551)

If you have the **9th edition**:

- **Ch. 7**: Sec. 7.1
  Sec. 7.2 (except Example 2)
  Sec. 7.3 (except formula (7) and Examples 1, 2, 5)
  Sec. 7.4 (except Theorem 7.4.4)
  Sec. 7.5
  Sec. 7.6
  Sec. 7.8
  Sec. 7.9 (only the method of variation of parameters on pages 436–438)
- **Ch. 8**: *Instead of the textbook, read pages 1–20 of the lecture notes posted online*
- **Ch. 9**: Sec. 9.1
  Sec. 9.2 (except pages 500–503)
  Sec. 9.3 (only page 511 on Jacobian matrix)
  Sec. 9.5 (from the beginning up to the end of Example 1 on page 537)

- Review homework questions in HWs # **4–6**.
- Practice the following study questions.

## I. System of ODEs

**1.** Determine whether or not the following matrices are non-singular. In the case they are non-singular, find their inverse $A^{-1}$.

$a)$ $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$  $b)$ $A = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$

**2.** Define linearly dependent and linearly independent vectors $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)}$ of size $n \times 1$.

**3.** Determine whether or not the following vectors are linearly independent:

$$\mathbf{y}^{(1)} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \qquad \mathbf{y}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

**4.** Determine the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

**5.** Write the following scalar ODE's as a system of 1st-order ODE's:

a) $y'' + t\,y' - 2\,y = 0$

b) $2\,y''' - t\,y' + y = \sin t$

**6.** First study the type of the following IVP's in terms of linearity/non-linearity (without proving). Then study the existence and uniqueness of the solution to the following IVPs, without solving them.

$a)$ ODE: $\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} \sin t & 2 \\ -1 & \frac{1}{t} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix}$,  IC: $\begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$b)$ ODE: $\begin{cases} y_1'(t) = y_1^2 + 2\,y_2 \\ y_2'(t) = t\,y_1 \end{cases}$  IC: $\begin{cases} y_1(0) = 0 \\ y_2(0) = 1 \end{cases}$

**7.** Determine whether or not $\mathbf{y}^{(1)} = [e^{-t},\, e^{-t}]^\top$ and $\mathbf{y}^{(2)} = [(t+1)\,e^{-t},\, (t+0.5)\,e^{-t}]^\top$ form a fundamental set of solutions of the ODE system:

$$\mathbf{y}' = P\,\mathbf{y}, \qquad P = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}.$$

**8.** Find the unique solution to the homogeneous ODE system

$$\text{ODE:} \quad \mathbf{y}' = P\,\mathbf{y}, \qquad\qquad \text{IC:} \quad \mathbf{y}(0) = [1,0]^\top,$$

for the following matrices:

*a)* $P = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$  *b)* $P = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

*c)* $P = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$  *d)* $P = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$

*e)* $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$  *f)* $P = \begin{bmatrix} 2 & 2 \\ -4 & -2 \end{bmatrix}$

*g)* $P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$  *h)* $P = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$

**9.** Find the unique solution to the non-homogeneous ODE system

$$\text{ODE:} \quad \mathbf{y}' = P\,\mathbf{y} + \mathbf{g}(t), \qquad\qquad \text{IC:} \quad \mathbf{y}(0) = [1,0]^\top,$$

for $\mathbf{g} = [e^{-t}, 0]^\top$ and with the matrices $P$ in question **8.**

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## II. Numerical ODEs

**10.** What is meant by *convergence* and *accuracy* of a numerical method?

**11.** Name two main advantages of RK4 over RK1. Motivate your answer.

**12.** Write the Euler's formula for the following IVP:

$$\begin{cases} y' = t\,y^2 + \cos(\pi t), & t \in [0,1] \\ y(0) = 1 \end{cases}$$

**13.** Write RK2 and RK4 formulas for the IVP in question **12**.

**14.** Find the approximate solution of the IVP in question **12** at $t = 1$, using RK1 and only two subintervals.

**15.** What is meant by *amplification factor*, and state condition that guarantees the stability of a numerical method.

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## III. Non-linear ODEs

**16.** Consider the following non-linear ODE system:

$$\begin{cases} y_1' = 1 + 2\,y_2 \\ y_2' = e^{2\,y_1} - 1 \end{cases}$$

a) Find an equation which implicitly gives the trajectories of the system.

b) Find the critical points of the system.

c) Find a linear system that approximates the non-linear system near the critical points.

d) Determine the type of the critical points by studying the linear systems obtained in part c.

**17.** Repeat parts a–d in question **16** for the following non-linear ODE system:

$$\begin{cases} y_1' = y_1 - y_1\,y_2 \\ y_2' = -y_2 + y_1\,y_2 \end{cases}$$

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**Good luck!**