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## Computation of the starting vortex flow past a flat plate

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**Abstract**

This paper compares two numerical methods applied to compute the starting vortex flow past a flat plate. The plate is inclined relative to a constant background flow at angle  $\alpha$ , with  $\alpha = 90^\circ, 60^\circ, 30^\circ$ . The numerical methods considered are (1) direct numerical simulation of the viscous flow (DNS), and (2) an inviscid vortex sheet model. The viscous DNS solves the Navier-Stokes equations by an operator splitting finite-difference method, for Reynolds numbers  $Re = 250, 500, 1000, 2000$ . The inviscid flow is computed by a regularized vortex sheet method, with the unsteady Kutta condition imposed at the edges of the plate, for regularization parameters  $\delta = 0.2, 0.1, 0.05$ . We present viscous vorticity contours, and compare streaklines and shed circulation obtained with both methods. Good agreement is found in the large-scale features of the separated spiral streaklines and the shed circulation as  $Re$  increases and  $\delta$  decreases. For small inclination angle  $\alpha$ , secondary separation on the downwind side of the plate introduces small-scale features in the viscous flow that are absent in the inviscid model. The vortex sheet model is much less costly than the viscous DNS, but it is limited by the omission of the boundary layers present in the viscous flow.

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**1. Introduction**

When a viscous fluid passes by a solid body of finite length, the flow sticks to the body and a boundary layer forms along the body surface. The boundary layer thickens due to viscous diffusion, and it is convected downstream until it separates and rolls up into a starting vortex. Depending on the inclination angle, starting vortices may be present at the leading and trailing edges of the body, and they affect the resulting lift and drag forces, an important issue in fields such as aerodynamics and biolocomotion<sup>1</sup>. Several groups have compared different models and numerical methods for computing these flows. For example a recent study compared a direct numerical simulation (DNS) with a discrete vortex model (DVM) for flow past an inclined airfoil<sup>9</sup>. The results showed that the DVM captures the large-scale features of the flow and was much less costly than the DNS, but the DVM was lacking in its treatment of the viscous boundary layer. Here we consider the starting vortex flow past a flat plate, inclined relative to a constant background flow at angle  $\alpha$ , and compare results obtained using DNS of the viscous flow and an inviscid regularized vortex sheet

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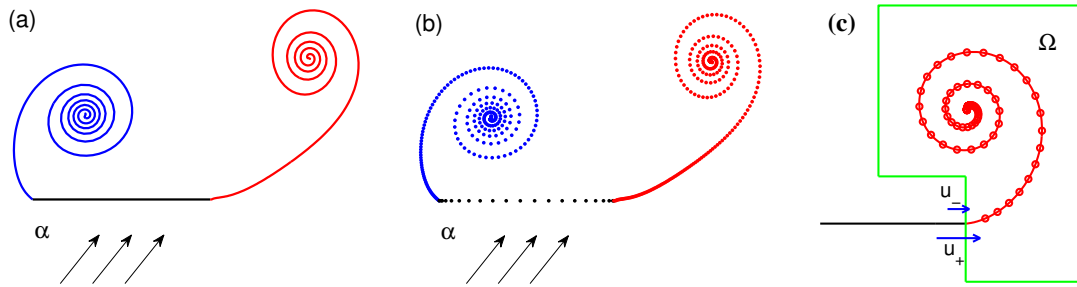


Fig. 1. (a) Vortex sheet model. (b) Discretization by point vortices. (c) Shed circulation and model for Kutta condition.

model. The vortex sheet model retains the cost advantage of a DVM, but is expected to better capture some details of the viscous flow. This was confirmed by Sheng et al.<sup>6</sup> who compared circulation and drag computed by DNS, DVM, and the vortex sheet model, for flow past a flapping plate. Here we compare streaklines and shed circulation for flow past a stationary inclined plate, for inclination angles  $\alpha = 90^\circ, 60^\circ, 30^\circ$ . The viscous DNS solves the Navier-Stokes equations by an operator splitting finite-difference method on a regular grid, using Crank-Nicolson for diffusion and a semi-Lagrangian method for convection, for Reynolds numbers  $Re = 250, 500, 1000, 2000$ . The inviscid flow is computed by a regularized vortex sheet method, with the unsteady Kutta condition imposed at the edges of the plate, for regularization parameters  $\delta = 0.2, 0.1, 0.05$ . We find generally good agreement between the viscous and inviscid models in the large-scale features of the flow. For large inclination angle  $\alpha$ , the streaklines and shed circulation agree well as  $Re$  increases and  $\delta$  decreases, while for small  $\alpha$ , secondary separation along the downwind side of the plate introduces small-scale structures in the viscous streaklines that are absent in the inviscid results.

The paper is organized as follows. Section 2 presents the problem and numerical methods. Section 3 presents the numerical results including vorticity contours for the viscous DNS, and comparison of the streaklines and shed circulation for the viscous DNS and inviscid regularized vortex sheet model, for increasing Reynolds numbers  $Re$  and decreasing regularization parameters  $\delta$ . The results are summarized in Section 4.

## 2. Problem Description and Numerical Method

### 2.1. Problem description

A flat plate of length  $L$  is immersed in a viscous fluid with kinematic viscosity  $\nu$ , in a uniform background potential flow of constant speed  $U$  inclined at angle  $\alpha$  to the plate (see figure 1(a)). We consider a range of angles  $\alpha = 90^\circ, 60^\circ, 30^\circ$ . The flow is described in Cartesian coordinates  $\mathbf{x} = (x, y)$  and time  $t$ , nondimensionalized with respect to the plate length  $L$  and background flow speed  $U$ . The plate is stationary, positioned horizontally on the  $x$ -axis with  $x \in [-0.5, 0.5]$ . The viscous flow is described by the incompressible Navier-Stokes equations given here in vorticity form,

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = Re^{-1} \nabla^2 \omega, \quad \nabla \cdot \mathbf{u} = 0, \quad (1)$$

where  $\omega$  is the scalar vorticity,  $\mathbf{u}$  is the fluid velocity, and  $Re = LU/\nu$  is the Reynolds number. The velocity is obtained from the stream function,  $\mathbf{u} = (\psi_y, -\psi_x)$ , where the vorticity and stream function are related by the Poisson equation,  $\nabla^2 \psi = -\omega$ , with boundary conditions  $\psi = 0$  on the plate and  $\psi \rightarrow \psi_\infty$  for  $|\mathbf{x}| \rightarrow \infty$ , where  $\psi_\infty$  is the background potential flow. The no-slip condition  $\mathbf{u} = 0$  holds on the plate.

### 2.2. DNS by an operator splitting finite-difference scheme

The Navier-Stokes equations are solved by operator splitting, where the inviscid and viscous parts are computed in two successive stages in each timestep. In the first stage, vorticity is updated on a rectangular grid with grid spacing  $\Delta x = \Delta y = h$ . The inviscid part,  $\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0$ , is computed by a semi-Lagrangian method, interpolating from

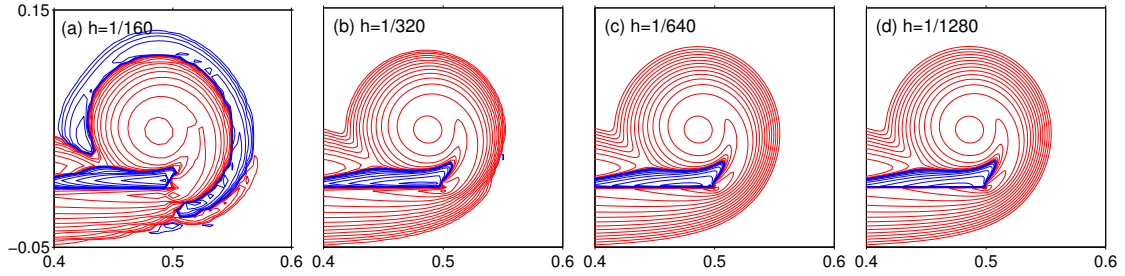


Fig. 2. Vorticity contours  $\omega = \pm 2^j$ ,  $j = -5, \dots, 10$ , obtained by DNS for  $Re = 500$  at  $t = 0.05$ , computed with (a)  $h = 1/160$ , (b)  $h = 1/320$ , (c)  $h = 1/640$ , (d)  $h = 1/1280$ . Positive vorticity is shown in red, negative in blue.

values at the previous time-step at the gridpoint pre-images under the flow map, and the viscous part,  $\partial_t \omega = Re^{-1} \nabla^2 \omega$ , is computed by Crank-Nicolson. In the second stage, the stream function is obtained from the updated vorticity by solving the Poisson equation using the conjugate gradient method. The far-field boundary condition is obtained numerically by evaluating an integral expression for  $\psi$ , and the no-slip condition is satisfied using Briley's formula; details are given in prior work<sup>4,7,8</sup>.

The effect of the grid spacing  $h$  is displayed in figure 2, showing vorticity contours near the edge of the plate at  $t = 0.05$ , for angle  $\alpha = 90^\circ$  and Reynolds number  $Re = 500$ , computed with decreasing  $h$ . At the time shown, upstream positive boundary layer vorticity (red) has been convected around the plate and forms a starting vortex on the downstream side of the plate. For the largest  $h$  shown (figure 2(a)), the vorticity is not well resolved, with a layer of spurious negative vorticity (blue) surrounding the vortex. In the results presented below, the grid spacing  $h$  is chosen sufficiently small so that the vorticity contours are well resolved and unchanged under further refinement, as in figure 2(c,d).

### 2.3. Vortex sheet model

The inviscid vortex sheet model is illustrated in figure 1(a). The vorticity distribution has two components; the separated vorticity is modelled by free vortex sheets emanating from the edges of the plate, shown in blue and red in figure 1(a), and the plate supports a bound vortex sheet whose strength  $\sigma(x)$  is determined by an integral equation ensuring that the normal fluid velocity vanishes on the plate. The free and bound vortex sheets are represented by points, as in figure 1(b). Specifically, the free sheets are represented by regularized point vortices (vortex blobs) with smoothing parameter  $\delta$ , and the bound sheet is represented by singular point vortices. At each timestep, new vortex blobs are released at the edges of the plate, as depicted in figure 1(c). The circulation shedding rate is

$$\frac{d\Gamma}{dt} = \frac{1}{2} (u_-^2 - u_+^2) = \bar{u} \cdot \sigma_e, \quad (2)$$

where  $u_-$ ,  $u_+$  are the fluid velocities on either side at the edge of the plate,  $\bar{u} = (u_- + u_+)/2$  is the average fluid velocity at the edge, and  $\sigma_e = u_- - u_+$  is the vortex sheet strength at the edge. The discussion on page 218 in Prandtl & Tietjens<sup>5</sup> shows that, assuming parallel flow, the circulation shedding rate (2) is independent of the fluid viscosity. Once released, the vortex blobs are convected by the free and bound vorticity with circulation invariant in time; details are given in prior work<sup>3</sup>.

## 3. Numerical Results

### 3.1. DNS vorticity contours for $Re = 500$

Figure 3 shows vorticity contours obtained from DNS with  $Re = 500$  at a sequence of times for  $\alpha = 90^\circ$  (left),  $\alpha = 60^\circ$  (middle), and  $\alpha = 30^\circ$  (right). The boundary layer vorticity is convected around the edges of the plate, from the upstream to the downstream side, forming starting vortices that grow and eventually move away from the plate. For  $\alpha = 90^\circ$ , the vortices are symmetric about the midplane  $x = 0$ , forming a counter-rotating vortex pair above the

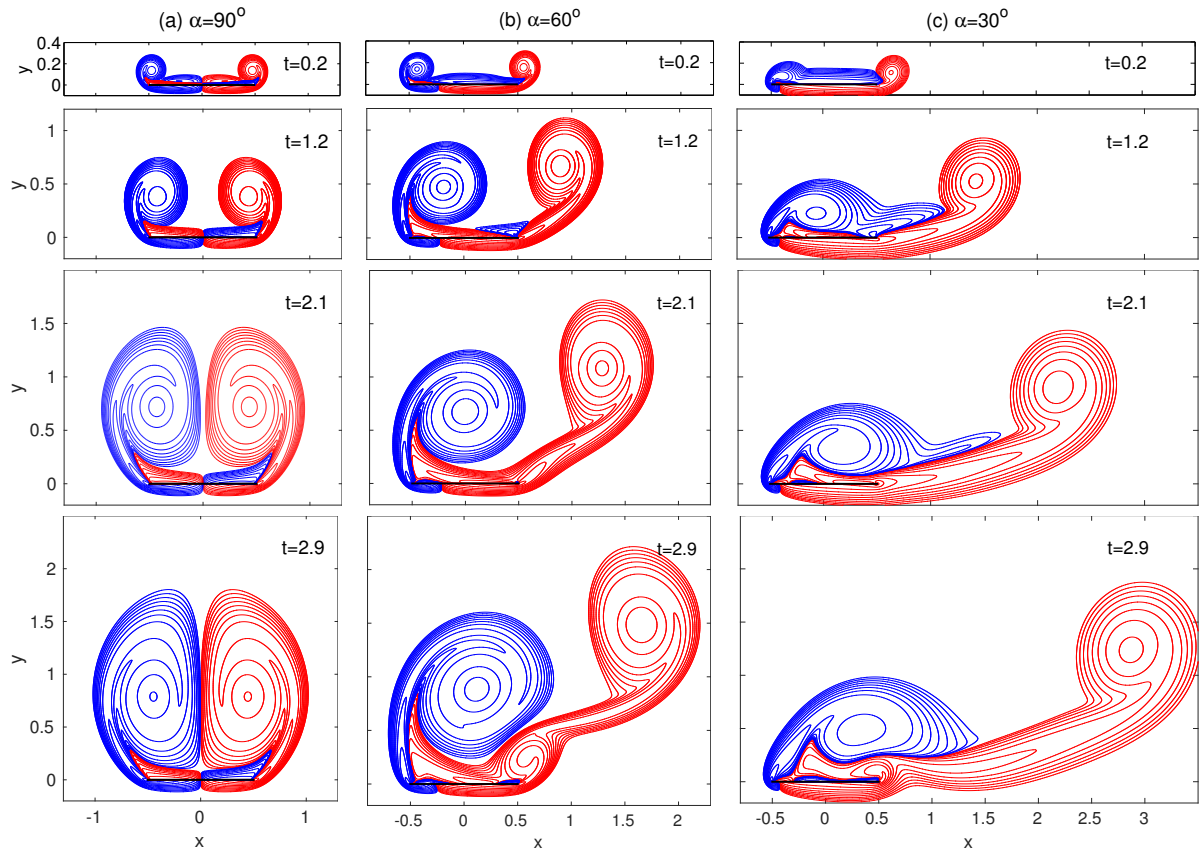


Fig. 3. Vorticity contours obtained from DNS with  $Re = 500$  at the indicated times, for  $\alpha = 90^\circ$  (left),  $60^\circ$  (middle),  $30^\circ$  (right).

plate. These vortices entrain opposite-signed boundary layer vorticity near the plate. For  $\alpha = 60^\circ$ , the symmetry is broken; positive boundary layer vorticity on the upstream side extends to  $x < 0$ , and negative boundary layer vorticity on the downstream side extends to  $x > 0$ . As time goes on, the positive starting vortex is convected away from the plate, while the negative starting vortex remains close is convected across the plate. This strengthens the downstream positive boundary layer vorticity, and a new positive starting vortex appears at the right edge at the final time. For  $\alpha = 30^\circ$ , the loss of symmetry is more pronounced, and although a new positive starting vortex is formed, it is weaker in this case.

### 3.2. Streaklines for $\alpha = 90^\circ, 60^\circ, 30^\circ$

Figures 4 and 5 present streaklines of the flow obtained from viscous DNS (figure 4) and the inviscid vortex sheet model (figure 5). In the viscous results in figure 4, the streaklines are obtained by inserting particles near the edges of the plate at each timestep, and tracking their evolution in the flow. These particles thereby mimic the dye particles that are released in experimental flow visualizations. For  $\alpha = 90^\circ$ , the streaklines form a pair of symmetric spirals centered on the starting vortices. For  $\alpha = 60^\circ$ , the left vortex remains closer and travels across the plate, while the right vortex is convected further away. At around time  $t = 2.1$ , the left vortex perturbs the right streakline, causing it to roll-up subsequently into a new starting vortex near the edge of the plate; this can be interpreted as a localized Kelvin-Helmholtz instability of the shear layer connecting the right starting vortex to the plate. For  $\alpha = 30^\circ$ , the left vortex stays closer to the plate, and although the right streakline is again perturbed and a new positive starting vortex forms, it appears to be weaker in this case, consistent with the results in figure 3(b,c).

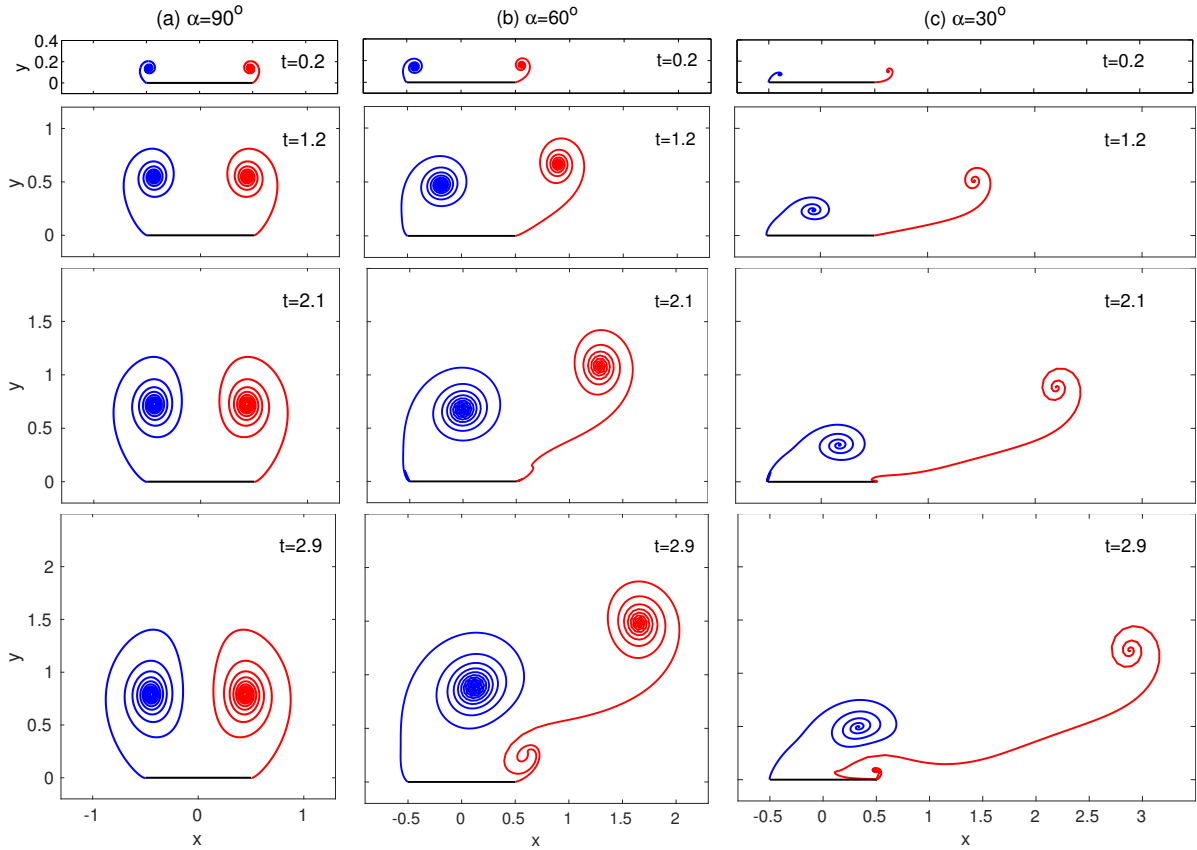


Fig. 4. Streaklines obtained from DNS with  $Re = 500$ , at the indicated times, for  $\alpha = 90^\circ$  (left),  $60^\circ$  (middle),  $30^\circ$  (right).

The streaklines in the vortex sheet results in figure 5 are obtained by connecting the vortex blobs that were released from the edges of the plate. Comparison of figures 4 and 5 shows good general agreement between viscous DNS and the inviscid vortex sheet model in terms of the size, shape, and location of the spirals. Similar results were previously obtained for the roll-up of a periodic free shear layer<sup>2</sup>, and for separation at the edge of a flapping plate<sup>6</sup>. For  $\alpha = 60^\circ$ , the present vortex sheet model captures the perturbation of the right streakline and the ensuing secondary spiral roll-up (figure 4(b), last time). The most notable difference between figures 4 and 5 is for  $\alpha = 30^\circ$ , at the final time, where the vortex sheet model does not capture the perturbation of the right streakline. The main reason for this difference is the presence of a strong boundary layer of positive vorticity above the plate in the viscous flow, visible in figure 3(c), which is absent in the vortex sheet model. This boundary layer convects the left vortex further to the right in the viscous case. As a result, the left vortex in the DNS is closer to the right streakline, enabling the perturbation, while this effect is absent in the vortex sheet model.

### 3.3. Separating streakline and shed circulation as $Re \rightarrow \infty$ and $\delta \rightarrow 0$

Figure 6 shows the dependence of the separating streakline on Reynolds number  $Re$  (DNS) and regularization parameter  $\delta$  (vortex sheet model). As  $Re \rightarrow \infty$  the viscous boundary layer thickness vanishes, and it may be expected that the regularized vortex sheet model recovers the inviscid (or zero viscosity) limit as  $\delta \rightarrow 0$ . Figure 6 shows that as  $Re \rightarrow \infty$  and  $\delta \rightarrow 0$ , the streakline rolls up more tightly in both cases, and the two models generally agree in terms of the size, shape, and location of the spiral vortex. However close inspection shows that the spirals are slightly wider for the vortex sheet model than for the viscous DNS, and the spiral center is shifted slightly to the right, and these

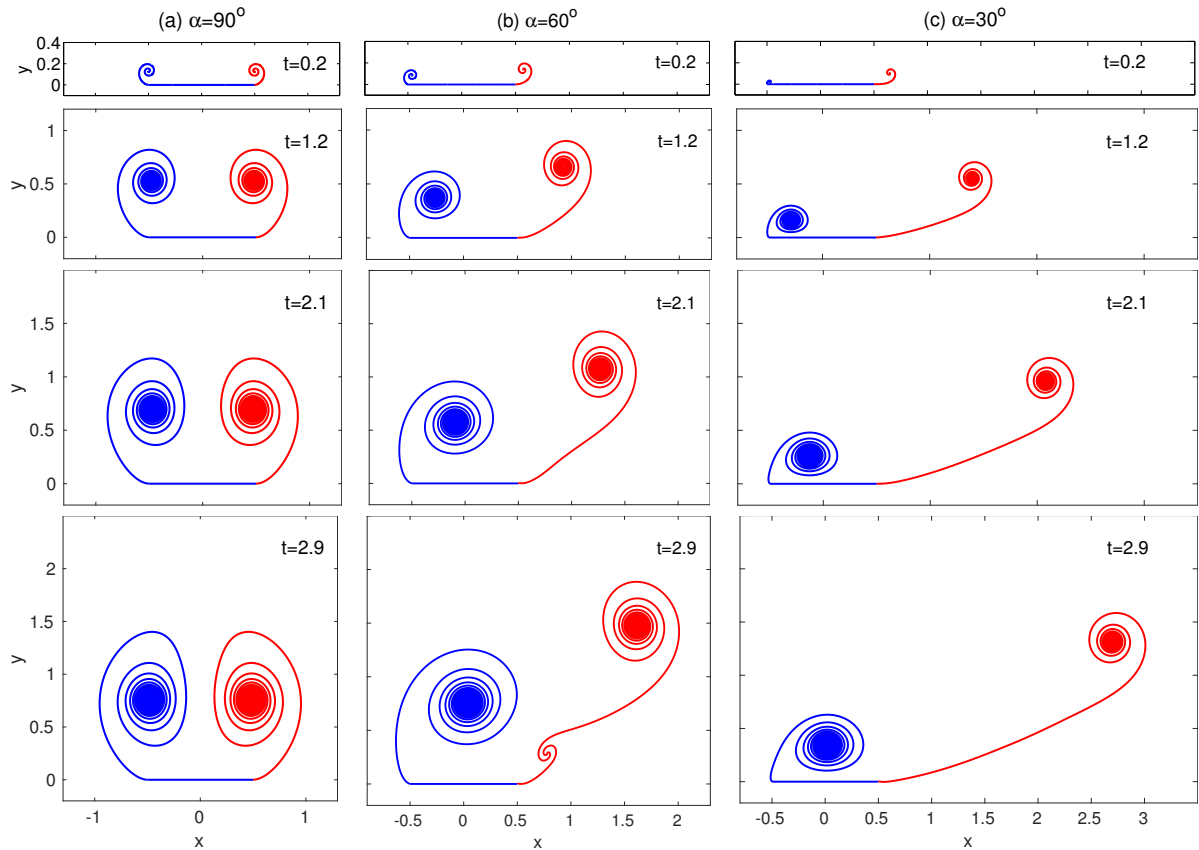


Fig. 5. Streaklines obtained from the vortex sheet model with  $\delta = 0.05$ , at the indicated times, for  $\alpha = 90^\circ$  (left),  $60^\circ$  (middle),  $30^\circ$  (right).

differences do not diminish over the range of  $Re$  and  $\delta$  used here. As shown next, these differences can be attributed to differences in the circulation shedding rate at early times, which cause the vortex sheet streakline to have more of its circulation concentrated in the spiral core, resulting in a slightly more radially symmetric structure.

The shed circulation is well defined in the vortex sheet model, but it requires definition in the case of the viscous flow. Figure 7 plots a closeup of the viscous vorticity at the right edge of the plate, at an early time, for the symmetric case  $\alpha = 90^\circ$ . Following previous work<sup>4,7,8</sup>, we define the shed circulation to be the area integral of the vorticity in the region  $\Omega(t)$  shown in the figure,  $\Gamma(t) = \int_{\Omega(t)} \omega(x, y, t) dA$ . This includes the vorticity in several places, (1) to the right of the vertical lines  $C^+$  and  $C^-$  through the edge of the plate, (2) above the curve  $C_0$  that separates the starting vortex from the boundary layer of opposite-signed vorticity above the plate, and (3) to the right of the slant line  $C^s$  passing through points at which the vorticity contours have maximal curvature (which we take as separating the starting vortex from the boundary layer further inboard along the plate). The remaining boundary of  $\Omega(t)$  is chosen to have negligible vorticity.

Figure 8(a) plots the shed circulation  $\Gamma(t)$  obtained from the viscous DNS with four values of the Reynolds number,  $Re = 250, 500, 1000, 2000$ , for  $t \in [0, 5]$ . The closeup shows that as  $Re$  increases, the circulation increases and appears to converge to a well defined limit. Figure 8(b) plots the corresponding results for the vortex sheet model with three values of the regularization parameter,  $\delta = 0.2, 0.1, 0.05$ . In this case the closeup shows that as  $\delta$  decreases, the circulation now decreases, yet it also appears to converge to a well defined limit. The agreement between the viscous DNS and the vortex sheet model in figure 8 is surprisingly good, especially near  $t = 5$  as seen in the closeups. A partial explanation was provided in prior work<sup>4</sup>; the main contribution to the viscous shed circulation is convection through

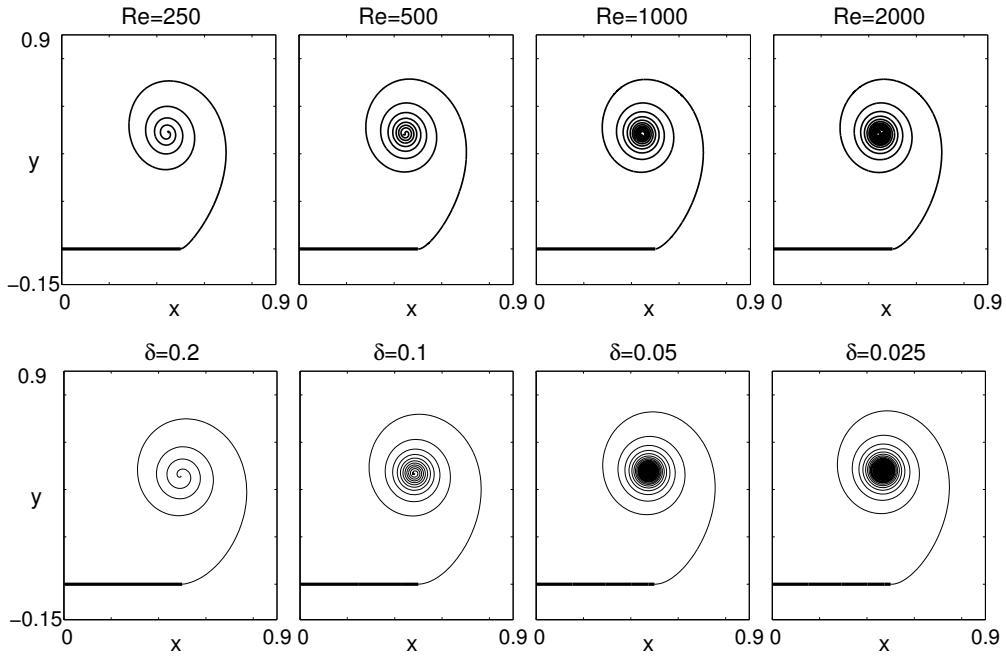


Fig. 6. Streaklines at  $t = 1$  for  $\alpha = 90^\circ$ . Top row (left to right): DNS with  $Re = 250, 500, 1000, 2000$ . Bottom row (left to right): Vortex sheet simulation with  $\delta = 0.2, 0.1, 0.05, 0.025$ .

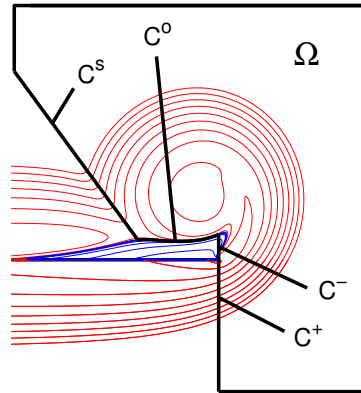


Fig. 7. Region  $\Omega(t)$  used to define the shed circulation  $\Gamma(t) = \int_{\Omega(t)} \omega dA$ .

$C^+$  and  $C^-$ , and this effect is captured by the unsteady Kutta condition in the vortex sheet model. However a closer look at figure 8(a,b) shows larger differences between the viscous and inviscid results at earlier times. For example at  $t = 1$ , the vortex sheet shed circulation is slightly larger, perhaps by 15%, than the viscous DNS shed circulation. This shows that the initial circulation shedding rate is larger for the inviscid vortex sheet model, which implies that the vorticity is more concentrated near the spiral core, thereby affecting the outer spiral turns of the streakline, as described earlier. A more detailed comparison of the viscous and inviscid shed circulation, for a range of incident angles  $\alpha$ , and accelerating background potential flows  $Ut^p$ , is currently in preparation.

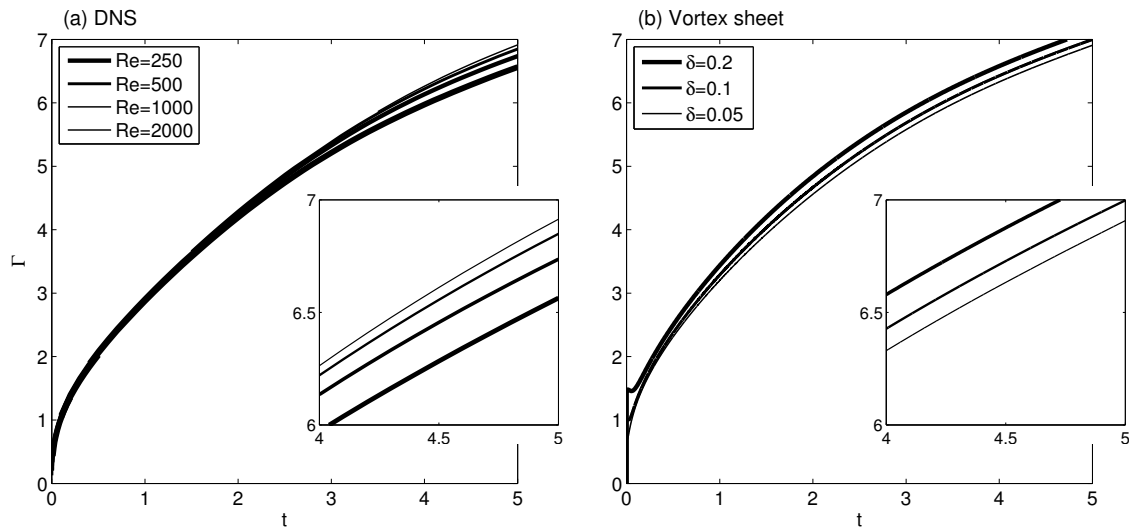


Fig. 8. Shed circulation  $\Gamma(t)$  for  $\alpha = 90^\circ$ . (a) DNS with  $Re = 250, 500, 1000, 2000$ . (b) Vortex sheet simulations with  $\delta = 0.2, 0.1, 0.05$ .

#### 4. Summary

The starting vortex flow at the edge of a flat plate inclined relative to a constant background flow, with inclination angle  $\alpha$ , was computed by direct numerical simulation (DNS) and a regularized vortex sheet model. The flow was computed for various values of  $\alpha$ , as well as a range of Reynolds numbers  $Re$  and regularization parameters  $\delta$ . Good agreement was found between the DNS and the vortex sheet model in terms of the large-scale features of the flow, namely the size, shape, and location of the spiral vortex cores, and the shed circulation as a function of time. The vortex sheet model is much less costly than the viscous DNS, but it is limited by the omission of the boundary layers present in the viscous flow. Hence there is a need for a viscous vortex method that addresses the limitations of the present vortex sheet model while retaining its computational cost advantage in simulating these fluid/structure interactions.

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