Fourier Series

Suppose $f(x)$ is a continuous function defined for $-\pi \leq x \leq \pi$. The *Fourier coefficients* of f are defined as follows:

$$a_n = \langle f(x), \cos(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx), \quad n \geq 0,$$
$$b_n = \langle f(x), \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx), \quad n \geq 1.$$

The *Fourier series* for f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

1. In this problem we will examine Fourier series for a few simple functions.

- **a.** Find the Fourier series for $f(x) = x$, $g(x) = x^2$, and $h(x) = |x|$.
- **b.** Graph the partial sums, for $f(x)$, $g(x)$, and $h(x)$, for $1, 2, 3, 4$, that is *truncate* the infinite sum in the Fourier expansions of $f(x), g(x), h(x)$ after the first, second, third, and fourth terms. Note, you can use a graphing calculator for this!
- **c.** What do you think happens to the graph of the *infinite sum*? For which values of x will it give back the function's value?
- **d.** Plug in $x = \frac{\pi}{2}$ to the Fourier expansion of x to obtain an infinite series expansion for π. How fast does this series converge to π (test by adding up the first few elements of the sequence)?

2. Find the Fourier series expansions for $f(x) = \cos(nx)$ for $n \geq 0$ and for $\sin(mx)$ for $m > 0$. Why are these so nice?

3. What are some natural phenomena for which you might expect Fourier series to be a particularly powerful tool of investigation?

4. Suppose f is twice differentiable with continuous second derivative on $[-\pi, \pi]$. Show that the Fourier series of f converges absolutely on $[-\pi, \pi]$ (that is, assume that the sum of the absolute values of sines and cosines in the Fourier series for f converges). Does it necessarily converge to f? How would you compare convergence properties of Fourier series with those of the more familiar Taylor series? Do Taylor series have some advantages over Fourier series?