Parseval's Formula

We will now address the situation of when the value of a function f can be recovered from its Fourier series: in other words, when is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)?$$

Recall that if f is a continuous function on $[-\pi, \pi]$ then

$$|f| = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2}.$$

1. Suppose $V = \mathbf{R}^3$ with the usual inner product.

- **a.** Show that $|v| = \sqrt{\langle v, v\rangle}$ is the usual Euclidean notion of size, namely the distance from v to $\vec{0}$.
- **b.** Which vector w, in the subspace generated by $\{e_1 = (1,0,0), e_2 = (0,1,0)\}$ is closest to v?
- **c.** How can you express w from problem **b** in terms of the inner product on $\mathbf{R}^3$?
- **d.** How does this relate to Problem 6 on the Inner Product problem set?

2. Let

$$v_k = f_k(x) = \frac{a_0}{2} + \sum_{n=1}^{k} a_n \cos{(nx)} + b_n \sin{(nx)}$$

so v_k is the truncated Fourier expansion of $f(x)$. Show that v_k is *the* vector, in the vector space of functions generated by $\{\cos{(0x)}, \ldots, \cos{(kx)}, \sin{(x)}, \ldots, \sin{(kx)}\}$, which is *closest* to f, that is which minimizes $|f - v_k|$.

3. Show that the best trigonometric approximation, relative to the distance $|\cdot|$, for $f(x)$ is the Fourier Series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Is the distance from f to its Fourier series necessarily zero? When the distance is zero does this mean that $f(x) = F(x)$ for all values of x? Suppose for the moment that $|f(x) - F(x)| = 0$. Give a plausible argument why

$$|f|^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \tag{1}$$

Is your argument rigorous? What convergence issues need to be ironed out? Similarly, if we have a second function $g(x)$ with convergent Fourier series

$$g(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx) + d_n \sin(nx)$$

then

$$\langle f(x), g(x) \rangle = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} a_n c_n + b_n d_n. \tag{2}$$

The equations (1) and (2) are referred to as *Parseval's Formula*.

4. It would be nice to have a large collection of functions for which one *knows* that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

According to previous work, two reasonable hypotheses in order to guarantee this would be:

a. $f(-\pi) = f(\pi)$,

b. f is twice differentiable with continuous second derivative. Show that the Fourier series for f does indeed converge under hypotheses **a** and **b**. In a situation where one *knows* that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

converges (and indeed converges *uniformly*) it follows that it is a continuous function. In this situation, Parsevals' formula says that

$$\left| f - \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right) \right| = 0.$$

If f is continuous, show that you can then conclude that $f = F$.

As you can see, the convergence issues which arise when studying Fourier series are quite subtle. Suppose f is a function which is nice enough so that its Fourier coefficients a_n and b_n are well defined and let F denote the corresponding Fourier series. Parseval's formula says that $|f - F| = 0$ which means that there is *no area* underneath the graph of $|f - F|^2$. This does not mean that $f - F = 0$ but it certainly means that it cannot be non–zero over an entire interval. One would like to somehow *identify* all functions which differ from one another only off of a "negligeable" set and then one could truly say that any $f = F$ in this new universe. In order to develop these ideas, one needs to study a little bit of measure theory and then introduce the space L^2 on the interval $[-\pi, \pi]$. The essential point to be gathered from this quick survey of Fourier analysis is that the functions $\sin(nx), \cos(mx)$ are very special periodic functions, special because they form a *basis* for all (sufficiently nice) periodic functions. Just as a vector in R^n can be prescribed by giving its coordinates with respect to the usual orthonormal basis, so a periodic function can be expressed as a sum of sine and cosine functions.