Symmetry for Functions

1. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a function. Some of the common types of symmetry for f are reflectional symmetry and translational symmetry.

a. Give some examples of functions with reflectional symmetry about the y–axis, that is functions for which $f(-x) = f(x)$ for all values of x.

b. Give some examples of functions with translational symmetry, that is functions for which there is a positive real number a so that $f(x) = f(x+a)$ for all values of x.

c. What does the graph of a function with translational symmetry look like? What about the graph of a function with reflectional symmetry? Which of these two symmetries is a more restrictive condition?

d. If a function f is symmetric about the line $y = x$ what can you say about f?

2. Here we will focus on translational symmetry and study the most familiar group of functions possessing this symmetry, namely the functions $\sin(mx)$ and $\cos(nx)$ where $m > 0$ and $n \geq 0$. The meaning of the computations that you make below will become clear in the next problem set when they will be related to the notion of inner product which we just studied. These computations are a bit repetitive so if this is not your cup of tea then just work a few to get an idea what is going on.

a. Show that $\int_{-\pi}^{\pi} \sin(m_1 x) \sin(m_2 x) = 0$ if $m_1 \neq m_2$.

b. Show that $\int_{-\pi}^{\pi} \cos(m_1 x) \cos(m_2 x) = 0$ if $m_1 \neq m_2$.

c. Show that $\int_{-\pi}^{\pi} \sin(mx) \cos(nx) = 0$ if $m > 0$ and $n \geq 0$.

d. Compute $\int_{-\pi}^{\pi} \sin(mx) \sin(mx)$ for $m \geq 1$.

e. Compute $\int_{-\pi}^{\pi} \cos(nx) \cos(nx)$ for $n \geq 0$.

3. Let $\mathcal{C}([-\pi, \pi])$ denote the set of continuous real valued functions on the interval $[-\pi, \pi]$.

a. Show that $\mathcal{C}([-\pi, \pi])$ is a real vector space.

b. Show that

$$\langle f, g \rangle = \int_{-\pi}^{\pi} fg$$

is an inner product on $\mathcal{C}([-\pi, \pi])$.

c. With respect to this inner product, what are $|\sin(mx)|$ and $|\cos(nx)|$? Do the functions $\sin(mx)$ for $m > 0$ and $\cos(nx)$ for $n > 0$ form an orthornormal set of vectors in $\mathcal{C}([-\pi, \pi])$? How can you fix this? What if we allowed $n = 0$? How can you fix this?

4. Do you think, by analogy with Problem **4d** on the Inner Product worksheet, that

$$f(x) = \frac{\langle f(x), \cos(0x)\rangle \cos(0x)}{2} + \sum_{n>0} \langle f(x), \cos(nx)\rangle \cos(nx) + \langle f(x), \sin(nx)\rangle \sin(nx),$$

assuming that you have fixed the inner product so that the sine and cosine functions are an orthonormal set of vectors?

a. What would equality *mean* in the above equation?

b. Why would this equality be such a strong and surprising result?

c. Are there any simple *necessary* conditions you can put on $f(x)$ in order for the equality to hold for all $-\pi \leq x \leq \pi$?