Isometries

An isometry of $\mathbf{R}^n$ is a map $f : \mathbf{R}^n \to \mathbf{R}^n$ which *preserves* distances, that is

$$d(x, y) = d(f(x), f(y))$$

for any two points $x, y \in \mathbf{R}^n$. Unless otherwise stated, the distance function is the normal euclidean distance, that is

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

An isometry is a very special type of symmetry.

1. Give some examples of isometries of $\mathbf{R}^1$, $\mathbf{R}^2$, $\mathbf{R}^3$. Do you think your list of examples is complete? How could you check?

2. Suppose $f : \mathbf{R}^n \to \mathbf{R}^n$ is an isometry with the property that $f(\vec{0}) = \vec{0}$.

- **a.** Show that f is a *linear* map, that is that $f(x+y) = f(x) + f(y)$ for any two vectors x, y and $f(ax) = af(x)$ for any real number a and any vector x (Hint: show that f is determined by the images of the unit coordinate vectors– then conclude that f is linear).
- **b.** What can you say about the matrix which represents f when $f(\vec{0}) = \vec{0}$ (i.e. is it invertible, what is its determinant,...)?
- **c.** How many degrees of freedom are there in choosing f? The set of matrices representing these maps is called the *special orthogonal group* and is denoted $SO(n)$: here you have found the dimension of this space.

3. Suppose we identify $\mathbf{R}^n$ with the subset of $\mathbf{R}^{n+1}$ consisting of vectors $(x_1, \ldots, x_n, 1)$ whose last coordinate is 1. Consdier $n+1$ by $n+1$ matrices of the form

$$\begin{pmatrix} M & v \\ \vec{0} & 1 \end{pmatrix}$$

where M is an orthonormal n by n matrix, v is a (column) vector in $\mathbf{R}^n$, and $\vec{0}$ denotes the zero (row) vector in $\mathbf{R}^n$.

a. Show that each of these matrices induces a symmetry of $\mathbf{R}^n$, with the given identification of $\mathbf{R}^n$ with a subset of $\mathbf{R}^{n+1}$.

b. Show that the map in part **a** gives a one–to–one correspondence between symmetries of $\mathbf{R}^n$ and a set of matrices.

c. Check that if M_1 and M_2 are any two matrices of this special form then $M_1 M_2$ still has this form. Why does this *have* to be true, in light of part **b** of this problem?

d. For $w \in \mathbf{R}^n$ let $t_w : \mathbf{R}^n \to \mathbf{R}^n$ be translation by the vector w. Which matrix M represents this symmetry?

4. Consider the following distance function on $\mathbf{R}^2$:

$$\delta((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

a. What are the isometries of $\mathbf{R}^2$ with respect to δ?

b. How does this compare to the isometries of $\mathbf{R}^2$ relative to the euclidean distance function?

5. Another distance function on $\mathbf{R}^2$ is the so–called discrete distance function

$$\delta'((x_1, x_2), (y_1, y_2)) = \begin{cases} 0 & \text{if } (x_1, x_2) = (y_1, y_2) \\ 1 & \text{if } (x_1, x_2) \neq (y_1, y_2) \end{cases}$$

a. What are the isometries of $\mathbf{R}^2$ with respect to δ'?

b. How does this compare to the isometries of $\mathbf{R}^2$ relative to the euclidean distance function?