Classifying Isometries of the Plane

1. Suppose $f : \mathbf{R}^2 \to \mathbf{R}^2$ is an isometry and let $\text{Fix}(f)$ denote the set of points fixed by f, that is

$$\text{Fix}(f) = \{x \in \mathbf{R}^2 : f(x) = x\}.$$

a. Show that $\text{Fix}(f)$ is either empty, a single point, a line, or all of $\mathbf{R}^2$. The logic to be implemented here is the following: if f fixes two distinct points P, Q then f must fix the line L containing P and Q. Moreover, if f fixes three non–collinear points P, Q, R then f is the identity. The argument here works best if you start from the most restrictive condition (fixing three non–collinear points) and then pass to two distinct points.

b. What can you say about f if $\text{Fix}(f)$ is all of $\mathbf{R}^2$?

c. What can you say about f if $\text{Fix}(f)$ is a line?

d. What can you say about f if $\text{Fix}(f)$ is a point?

e. Give some examples of maps f where $\text{Fix}(f)$ is empty.

f. Can these results be generalized to maps $f : \mathbf{R}^n \to \mathbf{R}^n$?

2. Suppose f_1 and f_2 are two reflections.

a. Show that $f_1 \circ f_2$ is either a translation or a rotation: here $\circ$ denotes composition of maps.

b. How, in terms of f_1 and f_2, can you distinguish the two possible cases in **a**?

c. Can any translation or rotation always be expressed as a composition of two reflections?

3. Here we will use problems **1** and **2** to help classify all symmetries of $\mathbf{R}^2$ in terms of reflections.

a. Show that any symmetry with at least one fixed point can be expressed either as a single reflection or as a product of two reflections. How do you distinguish the two cases?

b. Suppose now that f is a symmetry of $\mathbf{R}^2$ without fixed points. Show that there is reflection r with the property that $r \circ f$ has a fixed point. Use **a** to conclude that every symmetry of $\mathbf{R}^2$ can be expressed as a product of at most three reflections. Is this expression unique?

4. Here we will discuss the *orientation* of a symmetry of $\mathbf{R}^2$. Suppose $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a symmetry.

a. Show that there is a translation $t_a : \mathbf{R}^2 \to \mathbf{R}^2$ where $t_a(x) = a + x$ for all $x \in \mathbf{R}^2$ so that $f = t_a \circ g$ and $g(\vec{0}) = \vec{0}$. Is the translation t_a unique?

b. Suppose that we represent g by a matrix M as on the previous problem set. Show that the determinant of M is either 1 or -1.

c. Show that if the determinant of M is -1 then the number of reflections required in order to express the transformation g must always be odd. If the determinant of M is 1 then the number of reflections required in order to express g must always be even.

d. How would you distinguish the "odd" symmetries from the "even" symmetries geometrically?