Harmonic Analysis: from Fourier to Haar

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CHAPTER 1

Fourier series: some motivation

In this book we discuss three types of Fourier analysis: first, *Fourier series*, in which the input is a periodic function on $\mathbb{R}$, and the output is a two-sided series where the summation is over $n \in \mathbb{Z}$; second, *finite Fourier analysis*, where the input is a vector of length N with complex entries, and the output is another vector in $\mathbb{C}^N$; and third, the *Fourier transform*, where the input is a function on $\mathbb{R}$, and the output is another function on $\mathbb{R}$.

As an aside, we note that one can also do Fourier analysis on $\mathbb{R}^n$, on abelian groups, on graphs, and in still more abstract settings. See [**SS1**, Chapter 7]. Fourier analysis is also connected with representation theory.

1.1. Some examples and key definitions

The central idea of Fourier analysis is to break a function into a combination of simpler functions. We think of these simpler functions as *building blocks*. We will also be interested in reconstructing the original function from the building blocks. Here is a colorful analogy: *a prism or raindrop can break a ray of (white) light into all colors of the rainbow.* The analogy is quite apt: the different colors correspond to different *wavelengths/frequencies* of light. We will see that our simpler functions can also correspond to pure frequencies. For example, we will shortly consider sine and cosine functions of various frequencies as our first example of building blocks. When played aloud, a given sine or cosine function produces a *pure tone*, or note, or harmonic, at a single frequency. The term *harmonic analysis* evokes this idea of separation of a sound, or in our terms a function, into pure tones.

Expressing a function as a combination of building blocks is also called *decomposing* the function. In later Chapters we will study so-called *time–frequency decompositions*, in which each building block encodes information about time as well as about frequency, very much like musical notation does.

We begin with an example.

Example 1.1. (*Toy Model of Voice Signal*) Suppose Amanda is in Baltimore, and she calls her mother in Vancouver, saying 'Hi Mom, it's Amanda, and I can't wait to tell you what happened today.' What happens to the sound? As Amanda speaks, she creates waves of pressure in the air, which travel toward the phone receiver. The sound has duration, say about five seconds in this example, and intensity or loudness, which varies over time. It also has many other qualities that make it sound like speech rather than music, say. The sound of Amanda's voice becomes a *signal*, which travels along the phone wire or via satellite, and at the other end is converted back into a recognizable voice.

Let us try our idea of breaking Amanda's voice signal into simpler building blocks. Suppose the signal looks like the function $f(t)$ plotted in Figure 1.1. (In reality, her voice signal would be much more complicated.)

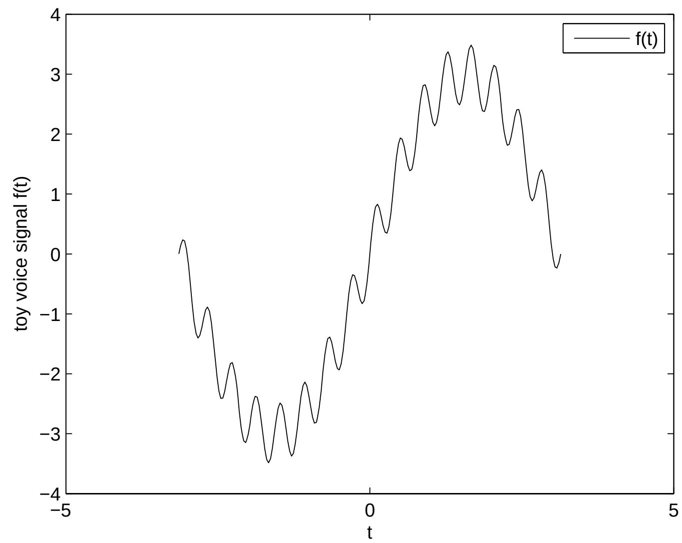

FIGURE 1.1. Toy voice signal $f(t)$.

In Figure 1.1 the horizontal axis represents time t in seconds, and the vertical axis represents the intensity of the sound, so that when $y = f(t)$ is near zero the sound is soft, and when $y = f(t)$ is large (positive or negative) the sound is loud. In this particular signal, there are two sorts of wiggling going on; see Figure 1.2.

We recognize the large, slow wiggle as a multiple of $\sin t$. The smaller wiggle is oscillating much faster; counting the oscillations, we recognize it as a multiple of $\sin 16t$. Here $\sin t$ and $\sin 16t$ are our first examples of building blocks. They are functions of *frequency* 1 and 16 respectively, meaning that they complete respectively 1 and 16 full oscillations as t runs through 2π units of time.

Next we need to know how much of each building block is present in our signal. The maximum amplitudes of the large and small wiggles are 3 and 0.5 respectively, giving us the terms $3\sin t$ and $\frac{1}{2}\sin 16t$. We add these together to build the original signal:

$$f(t) = 3\sin t + \frac{1}{2}\sin 16t.$$

We have written our signal $f(t)$ as a sum of constant multiples of the two building blocks. The right-hand side is our first example of a *Fourier decomposition*; it is the Fourier decomposition of our signal $f(t)$.

Let us get back to the phone company. When Amanda calls her mother, the signal goes from her phone to Vancouver. How will the signal be encoded? For instance, the phone company could take a collection of say 100 equally-spaced times, record the strength of the signal at each time, and send the resulting 200 numbers to Vancouver, where they will know how to put them back together. (By the way,

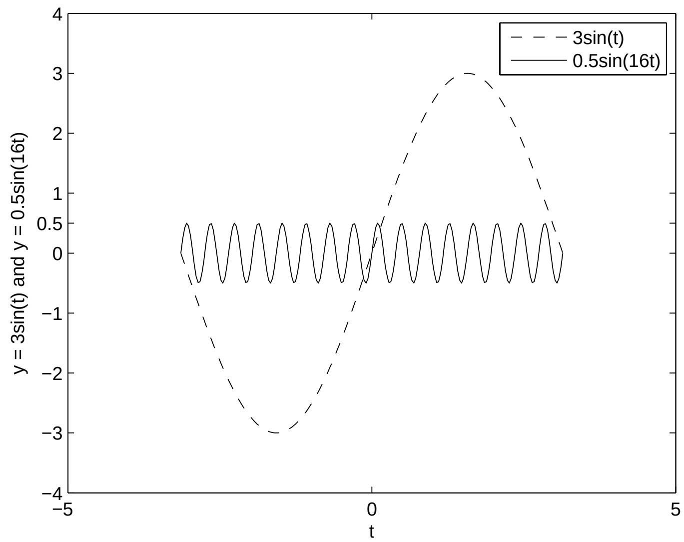

FIGURE 1.2. Building blocks of toy voice signal $f(t)$.

can you find a more efficient way of encoding the timing information?) But for our signal, we can achieve the same result with just *four* numbers. We simply send the frequencies 1 and 16, and the corresponding strengths or *amplitudes* 3 and 1/2. Once you know the code, that the numbers represent sine wave frequencies and amplitudes, these four numbers are all you need to know to rebuild our signal exactly.

Note that for a typical, more complicated signal, one would need more than just two sine functions as building blocks. Some signals would require infinitely many sine functions, and some would require cosine functions instead or as well. The collection

$$\{\sin nt : n \in \mathbb{N}\} \quad \cup \quad \{\cos nt : n \in \mathbb{N} \cup \{0\}\}$$

of all the sine and cosine functions whose frequencies are positive integers, together with the constant function with value 1, is an example of a *basis*. We will return to the idea of a basis later; informally, it means a given collection of building blocks that is able to express every function in some given class in a unique way. Fourier series use sines and cosines; other types of functions can be used as building blocks, notably in the wavelet series we will see later.

Now let us be more ambitious. Are we willing to sacrifice a bit of the quality of Amanda's signal in order to send it more cheaply? Maybe. For instance, in our signal the strongest component is the big wiggle. What if we send only the single frequency 1 and its corresponding amplitude 3? In Vancouver, only the big wiggle will be reconstructed, and the small fast wiggle will be lost from our signal. But Amanda's mother knows the sound of Amanda's voice, and the imperfect reconstructed signal may still be recognizable. This is our first example of *compression* of a signal.

FIGURE 1.3. Plot of an actual voice saying 'Hi Mom, it's Amanda, and I can't wait to tell you what happened today'. Explain the units: time in seconds along the horizontal axis, a proxy for volume/intensity along the vertical axis. Check how long it lasts and correct the estimate (currently 'five seconds') in the first paragraph of this example.

To sum up: We *analyzed* our signal $f(t)$, determining which building blocks were present and with what strength. We *compressed* the signal, discarding some of the frequencies and their amplitudes. We *transmitted* the remaining frequencies and amplitudes. (At this stage we could also have *stored* the signal.) At the other end they *reconstructed* the compressed signal.

Again, in practice one wants the reconstructed signal to be similar to the original signal. There are many interesting questions about which building blocks, and how many, can be thrown away while retaining a recognizable signal. This is part of the subject of signal processing, in electrical engineering.

In the interests of realism, Figure 1.3 shows a plot of an actual voice saying 'Hi Mom, it's Amanda, and I can't wait to tell you what happened today.' $\Diamond$

In another analogy, we can think of the decomposition as a recipe. The building blocks, distinguished by their frequencies, correspond to the different ingredients in your recipe. You also need to know *how much* sugar, flour, and so on you have to add to the mix to get your cake f; that information is encoded in the amplitudes or coefficients.

The function f in Example 1.1 is especially simple. The intuition that sines and cosines were sufficient to describe many different functions was gained from the experience of the pioneers of Fourier analysis who first tested these methods with physical problems (for instance heat diffusion and the vibrating string) in the early eighteenth century. This intuition leads to the idea of expressing a function $f(\theta)$ as an infinite linear combination of sines and cosines, also known as a *trigonometric series*:

$$f(\theta) \sim \sum_{n=0}^{\infty} \left[b_n \sin(n\theta) + c_n \cos(n\theta) \right]. \tag{1.1}$$

We use the symbol $\sim$ to indicate that the right-hand side is the trigonometric series *associated with* f. We don't use the symbol $=$ here since in some cases the left- and right-hand sides of (1.1) are not equal, as discussed below.

We can rewrite the right-hand side of (1.1) as a linear combination of exponential functions. To do so, we use *Euler's Formula*[1]: $e^{i\theta} = \cos\theta + i \sin\theta$, and the

[1] This formula is named after the Swiss mathematician Leonhard Euler (pronounced "oiler"), (1707–1783). We assume the reader is familiar with basic complex number operations and

corresponding formulas for sine and cosine in terms of exponentials, applied to $n\theta$ for each n,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

We will use the following version throughout:

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}. \tag{1.2}$$

The right-hand side of (1.2) is called the *Fourier series* of f. We also say that f can be *expanded* in a Fourier series. The coefficient a_n is called the n^{th} Fourier coefficient of f, and is often denoted by $\widehat{f}(n)$ to emphasize the dependence on the function f. The coefficients $\{a_n\}$ are determined by the coefficients $\{b_n\}$ and $\{c_n\}$ in the sine/cosine expansion.

EXERCISE 1.2. Write a_n in terms of the b_n and c_n. ◊

1.2. Main questions

Our brief sketch immediately suggests several questions.

- How can we find the coefficients a_n from the corresponding function f?
- Given the $\{a_n\}$, how can we reconstruct f?
- In what sense does the series converge? Pointwise? Uniformly? or yet in a different sense?
- Does it converge to f in any of the above senses? If it does, how fast does it converge?
- Which functions can we express with a trigonometric series

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\ ?$$

  For example, must f be Riemann integrable, or continuous?
- What other building blocks could we use besides sines and cosines, or equivalently exponentials?

We will develop answers to these questions in the following pages, but before doing that let us compare the above to the more familiar problem of power series expansions. An infinite differentiable function f can be expanded into a *Taylor series*[2] centered say at $x = 0$,

$$\text{Taylor series and coefficients:} \qquad \sum_{n=0}^{\infty} c_n\, x^n, \qquad \text{where } c_n = \frac{f^{(n)}(0)}{n!}. \tag{1.3}$$

The Taylor series always converges to $f(0) = c_0$ when evaluated at $x = 0$. In the 18^{th} century mathematicians discovered that the traditional functions of calculus ($\sin x$, $\cos x$, $\ln(1+x)$, $\sqrt{1+x}$, e^x, etc.) could be expanded in Taylor series and

notation. For example if $v = a + ib$, $w = c + id$, then $v + w = (a + c) + i(b + d)$ and $vw = (ac - bd) + i(bc + ad)$. The algebra is done as it would be for real numbers, with the extra fact that the imaginary unit i has the property that $i^2 = -1$. The *absolute value* of a complex number $v = a + ib$ is defined to be $|v| = \sqrt{a^2 + b^2}$. See [**Tao2**, Section 15.6] for a quick review of complex numbers.

[2]Named after the English mathematician Brook Taylor (1685–1731). Sometimes the Taylor series centered at $x = 0$ is called the Maclaurin series, named after the Scottish mathematician Colin Maclaurin (1698–1746).

that their series converge to the given functions in an open interval containing $x = 0$. These mathematicians were very good at manipulating power series and calculating with them. That led them to believe that the same would be true for all functions, which at the time meant infinitely differentiable functions. That dream was shattered by Cauchy's[3] discovery in 1821 of a counterexample.

EXAMPLE 1.3. (*Cauchy's Counterexample*) The function

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$$

is infinitely differentiable at every x, and $f^{(n)}(0) = 0$ for all $n \geq 0$. Thus its Taylor series is identically equal to zero. Therefore the Taylor series will converge to $f(x)$ only for $x = 0$. ◊

EXERCISE 1.4. Verify that Cauchy's function is infinitely differentiable. Concentrate on what happens at $x = 0$. ◊

The *Taylor polynomial of order* N (centered at 0) of a function f that can be differentiated at least N times is given by the formula

$$P_N(f,0)(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N. \tag{1.4}$$

EXERCISE 1.5. Verify that if f is a polynomial of order less than or equal to N, then f coincides with its Taylor polynomial of order N. ◊

A *trigonometric*[4] *polynomial of degree* M is a function of the form

$$f(\theta) = \sum_{k=-M}^{M} a_k e^{ik\theta},$$

EXERCISE 1.6. Verify that if f is a trigonometric polynomial then its coefficients $\{a_k\}$ can be calculated with the following formula:

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\, d\theta.$$

◊

In both the Fourier and Taylor series, the problem is how well and in what sense we can approximate a given function by using its Taylor polynomials (very specific polynomials) or by using its *partial Fourier sums* (very specific trigonometric polynomials).

1.3. Fourier series and Fourier coefficients

To begin answering our questions, the Fourier coefficients $\widehat{f}(n) = a_n$ are calculated using the formula suggested by Exercise 1.6:

$$\widehat{f}(n) = a_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta. \tag{1.5}$$

[3]The French mathematician Augustin Louis Cauchy (1789–1857).

[4]To be more precise, this particular trigonometric polynomial is 2π-periodic; see Section 1.3.1.

ASIDE 1.7. *The notation* $:=$ *indicates that we are defining the term on the left of the* $:=$*, in this case* a_n*. Occasionally we may need to use* $=:$ *instead, when we are defining a quantity on the right.*

One way to explain the appearance of formula (1.5) is to assume that the function f is equal to a trigonometric series,

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

and then to proceed formally[5], operating on both sides of the equation. Multiply both sides by an exponential function, and then integrate, taking the liberty of interchanging the sum and the integral:

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} a_n e^{in\theta}e^{-ik\theta}\,d\theta \\ &= \sum_{n=-\infty}^{\infty} a_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-ik\theta}\,d\theta \;=\; a_k. \end{aligned}$$

The last equality holds because

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-ik\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-k)\theta}\,d\theta = \delta_{n,k}, \tag{1.6}$$

where the *Kronecker*[6] *delta* $\delta_{n,k}$ is defined by

$$\delta_{n,k} = \begin{cases} 1, & \text{if } k = n; \\ 0, & \text{if } k \neq n. \end{cases} \tag{1.7}$$

We explore the geometric meaning of equation (1.6) in Chapter 5. In language we will meet there, the equation says that the exponential functions $\{e^{inx}\}_{n\in\mathbb{N}}$ form an *orthonormal set*.

ASIDE 1.8. *The usual rules of calculus apply to complex-valued functions. For example, if* u *and* v *are the real and imaginary parts of the function* $f : [a, b] \to \mathbb{C}$*, that is* $f = u + iv$ *where* u *and* v *are real-valued, then* $f' := u' + iv'$*. Likewise, for integration,* $\int f = \int u + i \int v$*. Here we are assuming that the functions* u *and* v *are differentiable in the first case, and integrable in the second.*

Notice that the minimal requirement on f for the Fourier coefficients (given by the integral in formula (1.5)) to exist is that the complex-valued function $f(\theta)e^{-in\theta}$ should be an integrable[7] function on $[-\pi, \pi)$. In fact, if $|f|$ is integrable, then so is

[5]Here the term *formally* means that we work through a computation 'following our noses', without stopping to justify every step. In this example we don't worry about whether the integrals or series converge, or whether it is valid to exchange the order of the sum and the integral. A rigorous justification of our computation could start from the observation that it is valid to exchange the sum and the integral if the Fourier series converges uniformly to f (see Chapters **??** and **??** for some definitions). Formal computations are often extremely helpful in building intuition.

[6]Named after Leopold Kronecker, German mathematician and logician (1823–1891).

[7]A function $g(\theta)$ is said to be *integrable* on $[-\pi, \pi)$ if $\int_{-\pi}^{\pi} g(\theta)\,d\theta$ is well-defined; in particular the value of the integral is a finite number. In harmonic analysis the notion of integral used is the *Lebesgue integral*. We expect the reader to be familiar with Riemann-integrable functions meaning that the function g is bounded and the integral exists in the sense of Riemann. Riemann-integrable

$f(\theta)e^{-in\theta}$ and

$$\begin{aligned}|a_n| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta\right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)e^{-in\theta}|\,d\theta && \text{(since } \left|\textstyle\int g\right| \leq \textstyle\int |g|) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|\,d\theta && \text{(since } |e^{-in\theta}| = 1) \\ &< \infty && \text{(since } f \text{ is integrable).}\end{aligned}$$

ASIDE 1.9. *It can be verified that the absolute value of the integral of an integrable complex-valued function is less than or equal to the integral of the absolute value of the function, the so-called* triangle inequality *for integrals,*

$$\left|\int f\right| \leq \int |f|.$$

This triangle inequality for integrals should not to be confused with the triangle inequality for complex numbers: $|a + b| \leq |a| + |b|$, *or the triangle inequality for integrable functions:* $\int |f + g| \leq \int |f| + \int |g|$. *They are all animals in the same family.*

We will spend some time exploring in what sense the Fourier series (1.2) approximates the original function f. We begin with some examples.

EXERCISE 1.10. Find the Fourier coefficients and the Fourier series for the trigonometric polynomial

$$f(\theta) = 3e^{-2i\theta} - e^{-i\theta} + 1 + e^{i\theta} - \pi e^{4i\theta} + \frac{1}{2}\,e^{7i\theta}.$$

◊

EXAMPLE 1.11. (*Periodic Ramp Function*) Compute the Fourier coefficients and the Fourier series for the function

$$f(\theta) = \theta, \qquad \text{for } -\pi \leq \theta < \pi.$$

The n^{th} Fourier coefficient is given by

$$\begin{aligned}\widehat{f}\,(n) = a_n &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}\,d\theta \\ &= \begin{cases}(-1)^{n+1}/(in), & \text{if } n \neq 0; \\ 0, & \text{if } n = 0.\end{cases}\end{aligned}$$

using integration by parts in the last line. Thus the Fourier series of $f(\theta) = \theta$ is given by

$$f(\theta) \sim \sum_{\{n\in\mathbb{Z}:n\neq 0\}} \frac{(-1)^{n+1}}{in} e^{in\theta} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin{(n\theta)}.$$

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functions are Lebesgue integrable. In Chapter 2 we will briefly review the Riemann integral. See [**Tao1**, Chapters 11], [**Tao2**, Chapters 19].

These integration methods are named after the French mathematician Henri Lebesgue (1875–1941), and the German mathematician Georg Friedrich Bernhard Riemann (1826–1866).

Note that both f and its Fourier series are *odd functions*. It turns out that the function and the series coincide for all $\theta \in (-\pi, \pi)$. At the endpoints $\theta = \pm\pi$ the series converges to zero (the midpoint of the jump), since $\sin(n\pi) = 0$ for all $n \in \mathbb{N}$. Notice that in this example the function and the series do not coincide at the endpoints where the jump discontinuity occurs. ♢

EXERCISE 1.12. Fill in the details in the integration-by-parts argument in the calculation in Example 1.11. ♢

1.3.1. 2π-Periodic functions. Note that we are considering functions

$$f : [-\pi, \pi) \to \mathbb{C}$$

that are complex-valued, and defined on a bounded half-open interval in the real line. We extend the function periodically to the whole of $\mathbb{R}$. For example, Figure 1.4 shows part of the graph of the periodic extension to $\mathbb{R}$ of the real-valued function f defined on $[-\pi, \pi)$ in Example 1.11.

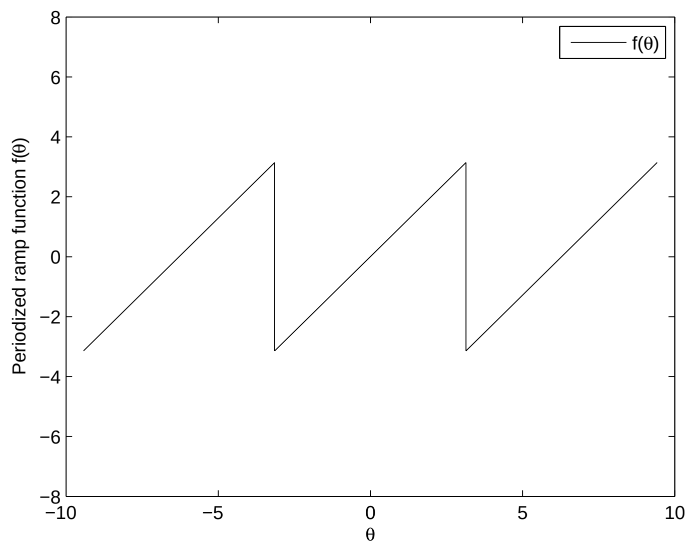

FIGURE 1.4. Periodized ramp function on $[-3\pi, 3\pi]$.

EXERCISE 1.13. Run the following MATLAB script to reproduce the plot in Figure 1.4.

```
clear all
plot([-3*pi,-pi],[-pi,pi],'k',[-pi,-pi],[-pi,pi],'k',...
  [-pi,pi],[-pi,pi],'k',[pi,pi],[-pi,pi],'k',[pi,3*pi],[-pi,pi],'k')
axis([-10,10,-8,8])
xlabel('\theta')
ylabel('Periodized ramp function f(\theta)')
legend('f(\theta)')}
```

◊

EXERCISE 1.14. Modify the MATLAB script from Exercise 1.13 to plot over $[-3\pi, 3\pi]$ the periodic extension to $\mathbb{R}$ of the function defined by $f(x) = x^2$ for $x \in [-\pi, \pi)$. ◊

In mathematical language, a function defined on $\mathbb{R}$ is 2π*-periodic* if for all $x \in \mathbb{R}$

$$f(x + 2\pi) = f(x).$$

The building blocks we are using are 2π-periodic functions: $\sin(n\theta)$, $\cos(n\theta)$, $e^{-in\theta}$. Finite linear combinations of 2π-periodic functions are 2π-periodic.

Let $\mathbb{T}$ denote the *unit circle*:

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} = \{z = e^{i\theta} : -\pi \leq \theta < \pi\}. \tag{1.8}$$

We can identify a 2π-periodic function f on $\mathbb{R}$ with a function $g : \mathbb{T} \to \mathbb{C}$ defined on the *unit circle* $\mathbb{T}$ as follows. Given the 2π-periodic function f and $\theta \in [-\pi, \pi)$ let $z = e^{i\theta}$, define g by

$$g(z) := f(\theta).$$

Note that given the 2π-periodic function f, for g to be continuous on the unit circle $\mathbb{T}$, we will need $f(-\pi) = f(\pi)$. When we say $f : \mathbb{T} \to \mathbb{C}$ and $f \in C^k(\mathbb{T})$, we mean that the function f is differentiable k times, and that f, f', $\dots$, $f^{(k)}$ are all continuous functions on $\mathbb{T}$. In words, *f is a k times continuously differentiable function from the unit circle into the complex plane.* In particular $f^{(\ell)}(-\pi) = f^{(\ell)}(\pi)$ for all $0 \leq \ell \leq k$.

Note that if f is 2π-periodic, then the integral of f over each interval of length 2π takes the same value. For example,

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = \int_{0}^{2\pi} f(\theta)\, d\theta = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} f(\theta)\, d\theta = \cdots .$$

EXERCISE 1.15. Verify that if f is 2π-periodic, then for all $a \in \mathbb{R}$

$$\int_{a}^{a+2\pi} f(\theta)\, d\theta = \int_{-\pi}^{\pi} f(\theta)\, d\theta.$$

◊

In general, if f is a *2π-periodic trigonometric polynomial of degree M*, in other words a function of the form

$$f(\theta) = \sum_{k=-M}^{M} a_k e^{ik\theta},$$

then its Fourier series coincides with f itself. See Exercises 1.6 and 1.17.

1.3.2. 2π- and L-periodic Fourier series and coefficients. So far, we have mostly considered functions defined on the interval $[-\pi, \pi)$ and their 2π-periodic extensions. Later we will also use functions defined on the unit interval $[0, 1)$, or a symmetric version of it $[-1/2, 1/2)$, and their 1-periodic extensions. More generally, we can consider functions defined on a general interval $[a, b)$ of length $L = b - a$, and extended periodically to $\mathbb{R}$ with period L.

An *L-periodic function* f defined on $\mathbb{R}$ is a function such that $f(x) = f(x + L)$ for all $x \in \mathbb{R}$. We can define the Fourier coefficients and the Fourier series for an

L-periodic function f. The formulae come out slightly differently to account for the rescaling:

$$L\text{-Fourier coefficients} \qquad \widehat{f}^{\,L}(n) = \frac{1}{L}\int_a^b f(\theta)e^{-2\pi in\theta/L}d\theta,$$

$$L\text{-Fourier series} \qquad \sum_{n=-\infty}^{\infty} \widehat{f}^{\,L}(n)e^{2\pi in\theta/L}.$$

In the earlier case $[a,b) = [-\pi,\pi)$, we had $2\pi/L = 1$. Note that the building blocks are now the L-periodic exponential functions $e^{2\pi in\theta/L}$, while the L-periodic trigonometric polynomials are *finite* linear combinations of these building blocks.

EXERCISE 1.16. Verify that for each $n \in \mathbb{Z}$, the function $e^{2\pi in\theta/L}$ is L-periodic. ♢

Often people consider 1-periodic functions. Their Fourier coefficients and series will read

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\,dx, \qquad \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx}.$$

Why is it important to consider functions defined on a general interval $[a,b)$? A key distinction in Fourier analysis is between functions f that are defined on a bounded interval, or equivalently periodic functions on $\mathbb{R}$, and functions f that are defined on $\mathbb{R}$ but that are not periodic. For non-periodic functions on $\mathbb{R}$, it turns out that the natural "Fourier quantity" is not a Fourier series but another function, known as the *inverse Fourier transform* of f. One way to develop and understand the Fourier transform is to consider Fourier series on symmetric intervals $[-L/2, L/2)$ of length L, and then let $L \to \infty$.

EXERCISE 1.17. Let f be an L-periodic trigonometric polynomial of degree N, that is

$$f(\theta) = \sum_{n=-M}^{M} a_n e^{2\pi in\theta/L}.$$

Verify that f coincides with its L-Fourier series. ♢

1.4. A little history, and motivation from the physical world

'It was this pliability which was embodied in Fourier's intuition, commonly but falsely called a theorem, according to which the trigonometric series *"can express any function whatever between definite values of the variable."* This familiar statement of Fourier's "theorem," taken from Thompson and Tait's "Natural Philosophy," is much too broad a one, but even with the limitations which must to-day be imposed upon the conclusion, its importance can still be most fittingly described as follows in their own words: The theorem *"is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly recondite question [sic] in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance."* '

Edward B. Van Vleck
Address to the American Association
for the Advancement of Science, 1913.
Quoted in [**Bre**, p. 12].

For the last 200 years, Fourier Analysis has been of immense practical importance, both for theoretical mathematics (especially in the subfield now called

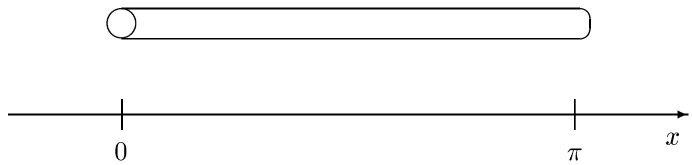

FIGURE 1.5. Sketch of a one-dimensional rod.

harmonic analysis), and in understanding, modeling, predicting, and controlling the behavior of physical systems. Older applications in mathematics, physics and engineering include the way that heat diffuses in a solid object, and the wave motion undergone by a string or a two-dimensional surface such as a drumhead. Recent applications include telecommunications (radio, wireless phones) and more generally data storage and transmission (the JPEG format for images on the Internet).

The mathematicians involved in the discovery, development and applications of trigonometric expansions include, in the 18th, 19th, and early 20th centuries: Brook Taylor (1685–1731), Daniel Bernoulli (1700–1782), Jean Le Rond d'Alembert (1717–1783), Leonhard Euler (1707–1783) (especially for the vibrating string), Joseph Louis Lagrange (1736–1813), Jean Baptiste Joseph Fourier (1768–1830), Peter Gustave Lejeune Dirichlet (1805–1859), and Charles De La Vallée Poussin (1866–1962), among others. See the books by Ivor Grattan-Guinness [**Grat**, Chapter 1] and by Thomas Körner [**Kor**] for more on the historical development.

We describe briefly some of the physical models that these mathematicians tried to understand, and whose solutions led them to believe that generic functions should be expandable in trigonometric series.

1.4.1. Temperature distribution in a one-dimensional rod. Our task is to find the temperature $u(x,t)$ in a rod of length π, where $x \in [0,\pi]$ represents the position of a point on the rod, and $t \in [0,\infty)$ represents time. See Figure 1.5. We assume that the initial temperature (when $t=0$) is given by a known function $f(x) = u(x,0)$, and that at both endpoints of the rod the temperature is held at zero, giving the boundary conditions $u(0,t) = u(\pi,t) = 0$. There are many other plausible boundary conditions.

The physical principle governing the diffusion of heat is expressed by the *Heat Equation*:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{1.9}$$

We want $u(x,t)$ to be a solution of the linear partial differential equation (1.9), with initial and boundary conditions

$$u(x,0) = f(x), \qquad u(0,t) = u(\pi,t) = 0. \tag{1.10}$$

We refer to equation (1.9) together with the initial conditions and boundary conditions (1.10) as an *initial value problem.*

As Fourier did in his *Théorie Analytique de la Chaleur* [**Fou**], we assume (or, more accurately, guess) that there will be a *separable* solution, meaning a solution that is the product of a function of x and a function of t:

$$u(x,t) = \alpha(x)\beta(t).$$

Then

$$\frac{\partial u}{\partial t}(x,t) = \alpha(x)\beta'(t), \qquad \frac{\partial^2 u}{\partial x^2} = \alpha''(x)\beta(t).$$

By equation (1.9), we conclude that

$$\alpha(x)\beta'(t) = \alpha''(x)\beta(t),$$

and so

$$\frac{\beta'(t)}{\beta(t)} = \frac{\alpha''(x)}{\alpha(x)}. \tag{1.11}$$

Because the left side of equation (1.11) depends only on t and the right side depends only on x, both sides of the equation must be equal to a constant. Call the constant k. We can decouple equation (1.11) into a system of two linear ordinary differential equations, whose solutions we know:

$$\begin{cases} \beta'(t) = k\beta(t) & \longrightarrow \quad \beta(t) = C_1 e^{kt}, \\ \alpha''(x) = k\alpha(x) & \longrightarrow \quad \alpha(x) = C_2 \sin(\sqrt{-k}\,x) + C_3 \cos(\sqrt{-k}\,x). \end{cases}$$

The boundary conditions tell us which choices to make:

$$\begin{aligned} \alpha(0) = 0 \quad &\longrightarrow \quad \alpha(x) = C_2 \sin(\sqrt{-k}\,x), \\ \alpha(\pi) = 0 \quad &\longrightarrow \quad \sqrt{-k} = n \in \mathbb{N}, \qquad \text{and thus } k = -n^2, \text{ with } n \in \mathbb{N}. \end{aligned}$$

We have found the following separable solutions of the initial value problem. For each $n \in \mathbb{N}$, there is a solution of the form

$$u_n(x,t) = Ce^{-n^2 t} \sin nx.$$

ASIDE 1.18. *A note on notation: Here we are using a typical convention from analysis, where C denotes a constant that depends on constants from earlier in the argument. In this case C depends on C_1 and C_2. An extra complication in this example is that C_1, C_2 and therefore C may all depend on the natural number n; it would be more illuminating to write*

$$u_n(x,t) = C_n e^{-n^2 t} \sin nx.$$

Finite linear combinations of these solutions will also be solutions of equation (1.9), since $\partial/\partial t$ and $\partial^2/\partial x^2$ are linear. We would like to say that *infinite* linear combinations

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$$

are also solutions.

To satisfy the initial condition $u(x,0) = f(x)$ at time $t = 0$, we require

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx.$$

So as long as we can write the initial temperature distribution f as a superposition of sine functions, we will have a solution to the equation. Here is where Fourier suggested that *all functions* have expansions into sine and cosine series. However, it was a long time before this statement was fully explored. See Section **??** for some highlights in the history of this quest.

1.4.2. Other physical models. In Table 1.1 we present schematically several physical models. In each case, one can do an heuristic analysis similar to the one performed in Section 1.4.1 for the heat equation in a one-dimensional rod.

EXERCISE 1.19. Find solutions to the physical models presented in Table 1.1 by an analysis similar to the one used in Section 1.4.1. ◊

The initial or boundary conditions in the table specify the value of u at time $t = 0$, that is $u(x, 0) = f(x)$.

TABLE 1.1. Physical models.

Model	Initial or Boundary Condition	Solution
Vibrating string $\boxed{u_{tt} = u_{xx}}$ $u(0,t) = u(\pi,t) = 0$	$f(x) = \sum_n b_n \sin nx$	$u(x,t) = \sum_n b_n e^{int} \sin nx$
Temperature in a rod $\boxed{u_t = u_{xx}}$ $u_x(0,t) = u_x(\pi,t)$	$f(x) = \sum_n c_n \cos nx$	$u(x,t) = \sum_n c_n e^{-n^2 t} \cos nx$
Steady-state temp. in a disk $\boxed{u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0}$	$f(\theta) = \sum_n a_n e^{in\theta}$	$u(r,\theta) = \sum_n a_n r^{\lvert n \rvert} e^{in\theta}$
Steady-state temp. in a rectangle $\boxed{u_{xx} + u_{yy} = 0}$	$f_0(x) = \sum_k A_k \sin kx$ $f_1(x) = \sum_k B_k \sin kx$	$u(x,y) = \sum_k \left(\frac{\sinh[k(1-y)]}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx$

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