1. Suppose $E \subset \mathbb{R}$ and $f_n \to f$ uniformly on $E$. Prove that if each $f_n$ is uniformly continuous on $E$ then $f$ is uniformly continuous on $E$.

2. Prove the integral test: Suppose that $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if $f$ is improperly Riemann integrable on $[1, \infty)$; that is, if and only if
$$\lim_{b \to \infty} \int_{1}^{b} f(x) \, dx$$
exists and is finite.

**Hint:** begin by establishing the inequality $f(k + 1) \leq \int_{k}^{k+1} f(x) \, dx \leq f(k)$, then sum over all $k = 1, \ldots, n - 1$.

3. Let $f : [a, b] \to \mathbb{R}$ be a function that is twice continuously differentiable ($f \in C^2([a, b])$). Show that if $f$ has three distinct zeros, then $f''$, the second derivative of $f$, has at least one zero.

4. Let $f : [a, b] \to \mathbb{R}$ be a function which is bounded and Riemann integrable on $[c, b]$ for every $c \in (a, b)$. Prove that $f$ is Riemann integrable on all of $[a, b]$.

5. In a metric space $(X, \rho)$ is it possible to have two distinct points $x, y$ in $X$ such that $B(x, r) = B(y, r)$ for some $0 < r < \infty$? Is this possible when $X = \mathbb{R}$ and $\rho$ is the usual metric? (Here $B(x, r)$ denotes the open metric ball $\{z \in X : \rho(x, z) < r\}$.)
6. Give a counterexample to show that the change of variable formula

\[ \int_{\phi(E)} f(u) \, du = \int_E f(\phi(x)) \left| \det(\phi'(x)) \right| \, dx \]

can fail to hold for Jordan regions \( E \subset \mathbb{R}^2 \) if the assumption that \( \phi \) is one-to-one is removed even though \( \det \phi'(x) \neq 0 \) in \( E \).

**Hint:** try taking \( \phi(r, \theta) = (r \cos \theta, r \sin \theta) \).

7. Let \( \Omega \subset \mathbb{R}^3 \) be a closed region for which the divergence theorem applies. Suppose that \( \phi : \Omega \to \mathbb{R} \) is a \( C^2(\Omega) \) function which solves the Laplace equation

\[ \Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \]

for all \((x, y, z) \in \Omega\). Show that if \( \phi \) vanishes on \( \partial \Omega \), then \( \phi(x, y, z) = 0 \) for every \((x, y, z) \in \Omega\).

**Hint:** what is the divergence of the vector field \( \phi \nabla \phi \)?

8. Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a continuously differentiable function and \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear transformation. Suppose that there exists \( M > 0 \) such that \( |F(x) - T(x)| \leq M|x|^2 \) for every \( x \in \mathbb{R}^n \). Show that if \( T \) is invertible, then \( F \) is one-to-one in a neighborhood of the origin.

**Hint:** what is the derivative of \( G(x) = F(x) - T(x) \) at the origin?