Let \((X, d)\) be a metric space. Suppose \(E\) is a nonempty subset of \(X\). Define the distance from \(x \in X\) to \(E\) as
\[
\rho_E(x) = \inf_{z \in E} d(x, z).
\]

(a) Prove that \(\rho_E(x) = 0\) if and only if \(x \in \overline{E}\) (the closure of \(E\) in the metric \(d\)).

(b) Prove that \(\rho_E(x)\) is a uniformly continuous function on \(X\) by showing that
\[
|\rho_E(x) - \rho_E(y)| \leq d(x, y), \quad \text{for all } x, y \in X.
\]

2. Prove the following part of the root test: Suppose \(\{c_n\}_{n=1}^\infty\) is a sequence of real numbers satisfying
\[
\limsup_{n \to \infty} |c_n|^{1/n} < 1.
\]
Show that the series \(\sum_{n=1}^\infty c_n\) converges absolutely.

3. Suppose \(f : \mathbb{R} \to \mathbb{R}\) is a uniformly continuous function on all of \(\mathbb{R}\). Let \(\{y_n\}_{n=1}^\infty\) be a sequence of real numbers. For each \(n \in \mathbb{N}\) define a new function, \(f_n(x) := f(x + y_n)\), for all \(x \in \mathbb{R}\). If \(\lim_{n \to \infty} y_n = 0\) show that the sequence of functions \(\{f_n\}_{n=1}^\infty\) converges uniformly on \(\mathbb{R}\).

4. A real valued function \(f\) on \([0, 1]\) is said to be Hölder continuous of order \(\alpha\) if there is a positive constant \(C\) such that \(|f(x) - f(y)| \leq C|x - y|^{\alpha}\) for \(x, y \in [0, 1]\). For these functions, define
\[
\|f\|_\alpha = \max_{0 \leq x \leq 1} |f(x)| + \sup_{0 \leq x, y \leq 1, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.
\]
Suppose \(0 < \alpha \leq 1\) and \(\{f_n\}_{n=1}^\infty\) is a sequence of Hölder continuous functions of order \(\alpha\) satisfying \(\|f_n\|_\alpha \leq 1\) for all \(n\). Show that \(\{f_n\}_{n=1}^\infty\) is an equicontinuous sequence. Conclude that there exists a subsequence \(\{f_{n_k}\}_{k=1}^\infty\) which converges uniformly on \([0, 1]\).

5. Suppose that \(f\) is a continuously differentiable real valued function on \([0, 1]\) (i.e. \(f'\) exists and is continuous on \([0, 1]\)). Show that \(f\) is Hölder continuous of order 1 (see the definition above).
6. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x, y) = (e^x \cos y, e^x \sin y).$$

Prove that given any $(x_0, y_0) \in \mathbb{R}^2$, $F$ is one-to-one in a neighborhood of $(x_0, y_0)$. Show that however $F$ is not one-to-one on all of $\mathbb{R}^2$.

7. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be an affine transformation if it is defined by $F(x) = Ax + b$, where $A$ is a non-singular $n \times n$ matrix, $Ax$ denotes matrix vector multiplication, and $b \in \mathbb{R}^n$. Suppose $E \subset \mathbb{R}^n$ is a bounded open set and that $F$ is an affine transformation.

(a) Show that $\text{Vol}(F(E)) = |\det A| \times \text{Vol}(E)$ (where $\text{Vol}(B)$ denotes the volume/area of the set $B$).

(b) The centroid of $E$ is defined as the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ where

$$\bar{x}_i = \frac{1}{\text{Vol}(E)} \int_E x_i \, dx$$

where the integral on the right is to be interpreted as the integral of the function $g(x) = x_i$ over the region $E$. Show that $F(\bar{x})$ is the centroid of $F(E)$.

8. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable ($C^1$) vector field. Prove that the following two statements are equivalent

(a) Given any domain $E \subset \mathbb{R}^3$ satisfying the hypotheses of the Divergence theorem with boundary $S = \partial E$

$$\int_S F \cdot n \, dS = 0$$

where $dS$ denotes the usual surface measure, and $n$ is the outward normal to the surface $S$.

(b) The identity $\text{div} \, F = 0$ holds on all of $\mathbb{R}^3$. 