1. Let $C[0,1]$ denote the space of real-valued continuous functions on $[0, 1]$ and

$$
X = \{ f \in C[0, 1] : \max_{x \in [0,1]} |f(x)| \leq 1 \}
$$

equipped with the metric

$$
\rho(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|.
$$

Show that $(X, \rho)$ is not compact by constructing an infinite set in $X$ with no limit point.

2. Suppose $f$ is a real-valued differentiable function on $[a, b]$ such that $f'$ exists and is continuous on $[a, b]$. Given $\epsilon > 0$ prove that there exists a $\delta > 0$ such that

$$
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon
$$

whenever $t, x \in [a, b]$ are any two points satisfying $0 < |t - x| < \delta$ (in other words, $f$ is in some sense “uniformly differentiable”). Hint: use the mean value theorem.

3. Suppose $f_k : [a, b] \to \mathbb{R}$ is a sequence of Riemann integrable functions on $[a, b]$ such that the series $\sum_{k=1}^{\infty} f_k$ is uniformly convergent.

   (a) Show that $\sum_{k=1}^{\infty} f_k$ is Riemann integrable.
   (b) Show that moreover,

$$
\sum_{k=1}^{\infty} \int_{a}^{b} f_k(x) \, dx = \int_{a}^{b} \sum_{k=1}^{\infty} f_k(x) \, dx.
$$

4. Suppose $f$ is Riemann integrable on $[0, A]$ for all $0 < A < \infty$, that $\lim_{x \to \infty} f(x) = 1$, and $t > 0$. Prove that

$$
\lim_{t \to 0^+} \int_{0}^{\infty} te^{-tx} f(x) \, dx = 1.
$$
5. A set $\Omega \subseteq \mathbb{R}^n$ is said to be path connected if given any $x, y \in \Omega$, there exists a continuous map $\gamma : [0, 1] \rightarrow \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = y$ (in other words, given any two points in $\Omega$, there exists a “path” lying in the set which joins them). Suppose $\Omega \subseteq \mathbb{R}^n$ is an open, path connected set and that $F : \Omega \rightarrow \mathbb{R}^m$ is differentiable on $\Omega$.

(a) Show that if $F'(x) = 0$ for every $x \in \Omega$, then $F$ is constant.
(b) Show that if $\Omega$ is not path connected then the result in (a) is not necessarily true.

6. Consider the family of rotations in $\mathbb{R}^2$, that is, the set of linear transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose matrix takes the form

$$
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

for some $\theta \in \mathbb{R}$.

(a) Prove that rotations are volume preserving, that is, $\text{Vol} S = \text{Vol} T(S)$ for all Jordan regions $S$.

(b) Given a Jordan region $S$ with positive volume, define its centroid as the point $(\bar{x}_1, \bar{x}_2)$ such that

$$
\bar{x}_i = \frac{1}{\text{Vol}(S)} \int_S x_i \, dA, \quad i = 1, 2,
$$

where the integral on the right is to be interpreted as the integral of the function $g_i(x) = x_i$ over the region $S$. Suppose that $T(S) = S$ for every $S$, that is, $S$ is invariant under rotations. Prove that the centroid of $S$ is the origin.

7. Let $\mathbf{F}$ be a continuously differentiable vector field on $\mathbb{R}^3 \setminus \{0\}$ such that $\text{div} \mathbf{F}(x) = \frac{1}{|x|}$. Given $0 < c < d$, find a relationship between

$$
\int_{S^2_c} \mathbf{F} \cdot \mathbf{n} \, dS \quad \text{and} \quad \int_{S^2_d} \mathbf{F} \cdot \mathbf{n} \, dS
$$

where $S^2_r$ denotes the sphere of radius $r$ in $\mathbb{R}^3$ and $\mathbf{n}$ denotes the outward normal vector field to that sphere.