Department of Mathematics and Statistics
University of New Mexico

Real Analysis **Qualifying Exam** **January 2014**

Instructions: Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let X be any nonempty set. Suppose $f : X \to \mathbb{R}$ is a bounded function on X and denote

$$\sup_X f = \sup\{f(x) : x \in X\} \qquad \text{and} \qquad \inf_X f = \inf\{f(x) : x \in X\}.$$

Prove that

$$\sup_X f - \inf_X f = \sup\{|f(x) - f(y)| : x, y \in X\}.$$

2. Prove the following parts of the so-called "limit comparison theorem": Suppose $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ are both series with $a_k \geq 0$, $b_k > 0$ for every $k = 1, 2, 3, \ldots$ and that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L.$$

   (a) Prove that if $0 \leq L < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.

   (b) Prove that if $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ also diverges.

3. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Set $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.

4. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Using the result from problem #1, show that f^2 is also a Riemann integrable function by proving that for any $\varepsilon > 0$ there exists a partition P such that $U(P, f^2) - L(P, f^2) < \varepsilon$. You may not apply the theorem which states that the composition of a continuous function with an integrable function is integrable.

5. Let $R = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$.

   (a) A function $P : R \to \mathbb{R}$ is said to have *separated variables* if

   $$P(x, y) = \sum_{k=1}^{N} c_k f_k(x) g_k(y)$$

   for some scalars $c_k \in \mathbb{R}$ and functions f_k, g_k continuous on $[a, b]$ and $[c, d]$ respectively. Prove that if $h(x, y)$ is continuous on R, there exists a sequence P_n of functions with separated variables such that $P_n \to h$ uniformly on R as $n \to \infty$.

   (b) Use the previous part to show the following elementary version of Fubini's theorem: If h is continuous on R, then

   $$\int_a^b \left(\int_c^d h(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b h(x, y)\, dx \right) dy.$$

6. Let $E \subset \mathbb{R}^n$ be an open set and suppose $f : E \to \mathbb{R}$ is differentiable on its domain. Prove that if f has a local maximum at a point $x \in E$, then $Df(x) = 0$.

7. Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be of class C^1 (all partial derivatives exist and are continuous); suppose that $f(a) = 0$ and that $Df(a)$ has rank n. Show that if c is a point of $\mathbb{R}^n$ sufficiently close to 0, then the equation $f(x) = c$ has a solution.

8. Given $a, b > 0$, let E be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, that is,

   $$E = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

   Show that the area of E is πab in two ways:

   (a) By computing $\iint_E 1\, dA$ with a change of variables.

   (b) By Green's theorem.