1. Let $X$ be any nonempty set. Suppose $f : X \to \mathbb{R}$ is a bounded function on $X$ and denote 
\[ \sup_X f = \sup \{ f(x) : x \in X \} \quad \text{and} \quad \inf_X f = \inf \{ f(x) : x \in X \}. \]
Prove that 
\[ \sup_X f - \inf_X f = \sup \{ |f(x) - f(y)| : x, y \in X \}. \]

2. Prove the following parts of the so-called “limit comparison theorem”: Suppose $\sum_{k=1}^{\infty} a_k$, 
\[ \sum_{k=1}^{\infty} b_k \] 
are both series with $a_k \geq 0$, $b_k > 0$ for every $k = 1, 2, 3, \ldots$ and that 
\[ \lim_{k \to \infty} \frac{a_k}{b_k} = L. \]
(a) Prove that if $0 \leq L < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.
(b) Prove that if $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ also diverges.

3. Suppose $f$ is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Set 
\[ g(x) = f(x + 1) - f(x) \]. Prove that $g(x) \to 0$ as $x \to \infty$.

4. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Using the result from problem #1, show 
that $f^2$ is also a Riemann integrable function by proving that for any $\varepsilon > 0$ there exists a 
partition $P$ such that $U(P, f^2) - L(P, f^2) < \varepsilon$. You may not apply the theorem which states 
that the composition of a continuous function with an integrable function is integrable.
5. Let \( R = [a, b] \times [c, d] \) be a rectangle in \( \mathbb{R}^2 \).

(a) A function \( P : R \to \mathbb{R} \) is said to have \textit{separated variables} if

\[
P(x, y) = \sum_{k=1}^{N} c_k f_k(x) g_k(y)
\]

for some scalars \( c_k \in \mathbb{R} \) and functions \( f_k, g_k \) continuous on \([a, b]\) and \([c, d]\) respectively. Prove that if \( h(x, y) \) is continuous on \( R \), there exists a sequence \( P_n \) of functions with separated variables such that \( P_n \to h \) uniformly on \( R \) as \( n \to \infty \).

(b) Use the previous part to show the following elementary version of Fubini’s theorem: If \( h \) is continuous on \( R \), then

\[
\int_a^b \left( \int_c^d h(x, y) \, dy \right) \, dx = \int_c^d \left( \int_a^b h(x, y) \, dx \right) \, dy.
\]

6. Let \( E \subset \mathbb{R}^n \) be an open set and suppose \( f : E \to \mathbb{R} \) is differentiable on its domain. Prove that if \( f \) has a local maximum at a point \( x \in E \), then \( Df(x) = 0 \).

7. Let \( f : \mathbb{R}^{k+n} \to \mathbb{R}^n \) be of class \( C^1 \) (all partial derivatives exist and are continuous); suppose that \( f(a) = 0 \) and that \( Df(a) \) has rank \( n \). Show that if \( c \) is a point of \( \mathbb{R}^n \) sufficiently close to 0, then the equation \( f(x) = c \) has a solution.

8. Given \( a, b > 0 \), let \( E \) be the region bounded by the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), that is,

\[
E = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.
\]

Show that the area of \( E \) is \( \pi ab \) in two ways:

(a) By computing \( \iint_E 1 \, dA \) with a change of variables.

(b) By Green’s theorem.