1. Let $f(X) \in \mathbb{Q}[X]$ be a polynomial of degree $n$. Let $E$ be a splitting field of $f$ over $\mathbb{Q}$. Show that $[E : \mathbb{Q}] \leq n!$.

2. Suppose $K$ is a field of characteristic zero and $G$ a finite group of automorphisms of $K$. Let $K^G$ be the subfield of $K$ fixed by $G$. Show that $K/K^G$ is a Galois extension with Galois group $G$.

3. Suppose $A, B$ are $n$ by $n$ matrix with complex coefficients. Show that $AB - BA$ cannot be equal to the identity matrix.

4. Consider the derivative map $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $D(f(t)) = f'(t)$ and the multiplication by $t$ map $M : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $M(f(t)) = tf(t)$. (Here $C^\infty(\mathbb{R})$ denotes the vector space of $C^\infty$ functions $\mathbb{R} \to \mathbb{R}$.) Compute the eigenvalues (and the corresponding eigenvectors) of the maps $D, M,$ and $D \circ M - M \circ D$.

5. Let $M = \mathbb{C}(z)$ be the field of rational functions of $z$ with $\mathbb{C}$ coefficients. Show that the map $SL_2(\mathbb{C}) \rightarrow Aut_\mathbb{C}(M)$ given by

$$
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} \mapsto \sigma, \quad \sigma(f(z)) = f\left(\frac{az + b}{cz + d}\right)
$$

is a group homomorphism. Compute the kernel and the image of this homomorphism.

6. Compute the center of the symmetric group $S_n$, $n \geq 3$.

7. Prove that the group defined by generators $a, b$ and one relation $a^2 b^3 = e$ is infinite.

8. Prove that if $p$ is an odd prime number then the group of invertible elements in the ring $\mathbb{Z}/p^n\mathbb{Z}$ is cyclic.

9. Prove that the ring $\mathbb{Z}[\sqrt{-5}]$ is not principal.

10. Prove that the ring $\{ \frac{n}{m} ; n, m \in \mathbb{Z}, m \not\in 5\mathbb{Z}\}$ is local.