Instructions: There are six (6) problems on this examination. Work all problems.

1. (15 points) Consider the following ODE system
\[
\begin{align*}
x' &= -x^2 y + y - y^3 \\
y' &= -x^3 + x^5 + x^3 y^2
\end{align*}
\]
(a) Find all equilibrium points.
(b) Find linearized systems at points (0, 0), (1, 0) and \((1/\sqrt{2}, 1/\sqrt{2})\).
(c) Determine the type and stability of each of these 3 points for the respective linearized systems.

2. (25 points) Consider the following initial-value-problem (IVP)
\[
yy'' + (y')^2 + f(y) = 0, \quad y(t)|_{t=0} > 0, \quad y'(t)|_{t=0} > 0, \quad t \geq 0,
\]
where \(f(y)\) is a continuous function such that \(f(y) \leq -f_0 y^{2+\epsilon}, f_0 > 0\) and \(\epsilon > 0\), for all \(y\). Hint: use a change of variable to reduce the order of the equation.
(a) Prove that the solution of the IVP blows up in finite time (i.e. \(|y(t)| \to \infty\) for \(t \to T < \infty\)).
(b) Find estimate for the blow up time \(T\).
(c) Rewrite that IVP (by a change of variable) as a conservative system in a potential.

3. (15 points) Find similarity solutions to the problem
\[
u_t = u_{xx}, \quad t > 0, x > 0; \quad u(x, 0) = 0, \quad x > 0; \quad u(0, t) = 1, \quad t > 0.
\]
Namely, look for a solution \(U(x, t) = f(x/\sqrt{t})\) and reduce the problem to a boundary value problem for an ordinary differential equation for \(f(z)\), where \(z\) is the similarity variable.

4. (15 points) Solve the problem
\[
xu_x + u_y = 1, \quad x \in \mathbb{R}, \quad y > 0, \quad u(x, 0) = \exp(x).
\]

5. (15 points) Let \(U \subset \mathbb{R}^n\) be an open set. Show that a function \(v \in C^2(U)\) that satisfies the mean-value property
\[
v(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} v dS
\]
for each closed ball \(B_r(x)\) of radius \(r\) centered at \(x\) with \(B_r(x) \subset U\) is necessarily harmonic. In the above, \(\omega_n = n\pi^{n/2}/\Gamma(n/2 + 1)\) is the surface area of the unit sphere in \(\mathbb{R}^n\).
6. (15 points) Show that the Cauchy problem for Laplace's equation is ill-posed. Consider the problem in two dimensions

\[ \Delta u = 0, \quad -\infty < x < \infty, \quad y > 0, \]
\[ u(x, 0) = f(x), \quad u_y(x, 0) = g(x), \quad -\infty < x < \infty \]

Construct a sequence of separated solutions

\[ u_n(x, y) = \frac{1}{n} Y_n(y) \cos nx \]

such that \( u_n(x, 0) \to 0, \ (u_n)'(x, 0) \to 0 \) as \( n \to \infty \), while \( u_n(x, 1) \to \infty \). How can this be used to show that the solution does not depend continuously on the initial data?