Winter 2015

Instructions: There are six (6) problems on this examination. Work all problems for full credit.

1. (20 points) Find a general solution of the following ODE:

\[ t^2 y'' - t(t + 2)y' + (t + 2)y = \frac{6 - 6t + 2t^2}{t} e^{2t}, \]

given that \( y_1(t) = t \) is a solution of the corresponding homogeneous equation.

2. (10 points) For the nonlinear system:

\[
\begin{align*}
    x' &= x - y + x^2 - xy, \\
y' &= -y + x^2.
\end{align*}
\]

(a) determine the critical points for the equation;
(b) determine the linearized system for each critical point and discuss whether it can be used to approximate the behaviour of the non-linear system;
(c) what is the type and stability of each critical point?

3. (20 points) Show that the nonlinear system

\[
\begin{align*}
    \dot{x} &= 3x - y - 4x^3 - 7xy^2, \\
    \dot{y} &= x + 3y - 4x^2y - 7y^3,
\end{align*}
\]

has at least one periodic orbit in the annulus \( \sqrt{3/7} < r < \sqrt{3/2} \) (pay attention, that you have to show that trajectory cannot be on the boundary of the annulus!).
4. (25 points) Solve the following Cauchy problem for a first order PDE:

\[(2x_1 + x_2)u_{x_1} + (x_2 + 1)u_{x_2} = u^2, \quad u(x_1, x_2)|_{x_2=1} = x_1^2 + 1, \quad x_1 \geq 0, x_2 \geq 1\]

and find an implicit condition over \(x_1\) and \(x_2\) under which this Cauchy problem has a bounded solution.

5. (15 points) Use the separation of variables to solve the following nonhomogeneous initial/boundary-value problem

\[u_{tt} - 4u_{xx} = 8, \quad x \in (0, l), \quad t \in (0, \infty), \quad l > 0, \quad a = \text{const},\]

\[u|_{x=0} = u|_{x=l} = 0, \quad t \in (0, \infty),\]

\[u(x, t)|_{t=0} = x(l - x) + \sin \left( \frac{5\pi x}{l} \right),\]

\[u_t(x, t)|_{t=0} = 0.\]

Hint: represent solution as \(u = v + w\), where \(v\) is the solution of nonhomogeneous \(t\)-independent problem and \(w\) is the solution of \(t\)-dependent homogeneous (i.e. with zero right hand side) problem.

6. (10 points) Prove the uniqueness of the solution \(u(x, t)\) of the following initial/boundary value problem for the heat equation

\[u_t - u_{xx} = 0, \quad x \in (0, l), \quad t \in (0, \infty), \quad l > 0,\]

\[u_x|_{x=0} = 0, \quad u|_{x=l} = 0, \quad t \in (0, \infty),\]

\[u(x, t)|_{t=0} = u_0(x).\]