Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **August 2011**

Instructions: Hand in 7 out of the 9 following problems. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let f be the "ruler function" on $[0,1]$ given by $f(x) = 1/2^n$ when $x = p/2^n$, p is an odd integer and $f(x) = 0$ otherwise.
   (a) For which x is f continuous and discontinuous and why?
   (b) For which x does $f'(x)$ exist and why?

2. Show that the subset of the complex plane $S = \{e^{2\pi i/n} : n = 1, 2, 3, \dots\}$ is compact using the definition of compactness.

3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on $\mathbb{R}$. Define what it means for f_n to converge to f uniformly. Then prove that if $f_n \to f$ uniformly then f must be continuous.

4. Given a 2 dimensional vector (x_1, x_2), define its p-norm as
$$\|(x_1, x_2)\|_p = \begin{cases} (|x_1|^p + |x_2|^p)^{\frac{1}{p}} & 1 \le p < \infty \\ \max(|x_1|, |x_2|) & p = \infty \end{cases}$$
   (a) In the Euclidean plane, geometrically describe the "unit balls"
$$\{(x_1, x_2) : \|(x_1, x_2)\|_p \le 1\}$$
   in the $p = 1, 2, \infty$ norms.
   (b) Show that for a vector (x_1, x_2) its p-norm converges to its ∞-norm as $p \to \infty$. In other words, show that
$$\lim_{p \to \infty} \|(x_1, x_2)\|_p = \|(x_1, x_2)\|_\infty .$$

5. Let f be a continuous, real valued function on $[0,1]$ such that $\int_0^1 f(x)\, x^n\, dx = 0$ for any $n = 0, 1, 2, 3, \dots$. Show that $f(x) = 0$ for all $x \in [0,1]$.

6. Suppose Ω is a region in $\mathbb{R}^2$ which can be characterized in the following two ways
$$\Omega = \{(x,y) : u_1(x) \le y \le u_2(x), a \le x \le b\} = \{(x,y) : v_1(y) \le x \le v_2(y), c \le y \le d\}$$
   for some continuous functions u_1, u_2, v_1, v_2. Prove Green's theorem for Ω. That is, show that if $P(x,y)$ and $Q(x,y)$ are C^1 functions in a neighborhood of Ω, and C is the boundary of Ω then
$$\oint_C P\, dx + Q\, dy = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dxdy.$$

7. Suppose S is an orientable surface with a nonempty boundary curve C for which Stokes' theorem is valid for all C^1 vector fields. Suppose $\mathbf{F}, \mathbf{F}_k$ are C^1 vector fields such that $\mathbf{F}_k \to \mathbf{F}$ uniformly on C. Show that

$$\lim_{k\to\infty} \iint_S \textbf{curl } \mathbf{F}_k \cdot \mathbf{n}\, dS = \iint_S \textbf{curl } \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is a continuous normal vector field on S.

8. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable map such that the Jacobian determinant $\det(DF(x))$ is nonzero for every $x \in \mathbb{R}^n$. Show that

$$\lim_{r\to 0^+} \frac{\text{Vol}(F(B_r(x_0)))}{\text{Vol}(B_r(x_0))} = |\det(DF(x_0))|$$

for every $x_0 \in \mathbb{R}^n$.

9. Suppose $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is C^2 and that $f_y(0, 0) \neq 0,\ f(0, 0) = 0$.

   (a) Show that there exists a neighborhood $(-\epsilon, \epsilon)$ and a continuously differentiable, real valued function ϕ defined on this set such that $\phi(0) = 0$ and $f(x, \phi(x)) = 0$.

   (b) Show that the vector $\langle 1, \phi'(x)\rangle$ is orthogonal to the vector $\langle f_x(x, \phi(x)), f_y(x, \phi(x))\rangle$ for all $x \in (-\epsilon, \epsilon)$.

   (c) Now define the map $F(x, w) = (x + wf_x(x, \phi(x)), \phi(x) + wf_y(x, \phi(x)))$. Show that F is one-to-one in a neighborhood of the origin.