Real Analysis Qualifying Exam August 2011

Instructions: Hand in 7 out of the 9 following problems. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let \( f \) be the “ruler function” on \([0, 1]\) given by \( f(x) = 1/2^n \) when \( x = p/2^n \), \( p \) is an odd integer and \( f(x) = 0 \) otherwise.
   
   (a) For which \( x \) is \( f \) continuous and discontinuous and why?
   
   (b) For which \( x \) does \( f'(x) \) exist and why?

2. Show that the subset of the complex plane \( S = \{ e^{2\pi i/n} : n = 1, 2, 3, \ldots \} \) is compact using the definition of compactness.

3. Let \( \{f_n\}_{n=1}^\infty \) be a sequence of real valued functions defined on \( \mathbb{R} \). Define what it means for \( f_n \) to converge to \( f \) uniformly. Then prove that if \( f_n \to f \) uniformly then \( f \) must be continuous.

4. Given a 2 dimensional vector \((x_1, x_2)\), define its \( p \)-norm as
   
   \[
   \|(x_1, x_2)\|_p = \begin{cases} 
   \left(\left|x_1\right|^p + \left|x_2\right|^p\right)^{\frac{1}{p}} & 1 \leq p < \infty \\
   \max(|x_1|, |x_2|) & p = \infty
   \end{cases}
   \]

   (a) In the Euclidean plane, geometrically describe the “unit balls” \( \{(x_1, x_2) : \|(x_1, x_2)\|_p \leq 1\} \) in the \( p = 1, 2, \infty \) norms.

   (b) Show that for a vector \((x_1, x_2)\) its \( p \)-norm converges to its \( \infty \)-norm as \( p \to \infty \). In other words, show that
   
   \[
   \lim_{p \to \infty} \|(x_1, x_2)\|_p = \|(x_1, x_2)\|_\infty.
   \]

5. Let \( f \) be a continuous, real valued function on \([0, 1]\) such that \( \int_0^1 f(x) x^n dx = 0 \) for any \( n = 0, 1, 2, 3, \ldots \) Show that \( f(x) = 0 \) for all \( x \in [0, 1] \).

6. Suppose \( \Omega \) is a region in \( \mathbb{R}^2 \) which can be characterized in the following two ways
   
   \[
   \Omega = \{(x, y) : u_1(x) \leq y \leq u_2(x), a \leq x \leq b\} = \{(x, y) : v_1(y) \leq x \leq v_2(y), c \leq y \leq d\}
   \]

   for some continuous functions \( u_1, u_2, v_1, v_2 \). Prove Green’s theorem for \( \Omega \). That is, show that if \( P(x, y) \) and \( Q(x, y) \) are \( C^1 \) functions in a neighborhood of \( \Omega \), and \( C \) is the boundary of \( \Omega \) then
   
   \[
   \oint_C P \, dx + Q \, dy = \iint_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx \, dy.
   \]
7. Suppose $S$ is an orientable surface with a nonempty boundary curve $C$ for which Stokes’ theorem is valid for all $C^1$ vector fields. Suppose $F, F_k$ are $C^1$ vector fields such that $F_k \to F$ uniformly on $C$. Show that

$$\lim_{k \to \infty} \int_S \text{curl } F_k \cdot n \ dS = \int_S \text{curl } F \cdot n \ dS$$

where $n$ is a continuous normal vector field on $S$.

8. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable map such that the Jacobian determinant $\det(DF(x))$ is nonzero for every $x \in \mathbb{R}^n$. Show that

$$\lim_{r \to 0^+} \frac{\text{Vol}(F(B_r(x_0)))}{\text{Vol}(B_r(x_0))} = |\det(DF(x_0))|$$

for every $x_0 \in \mathbb{R}^n$.

9. Suppose $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is $C^2$ and that $f_y(0, 0) \neq 0, f(0, 0) = 0$.

(a) Show that there exists a neighborhood $(-\epsilon, \epsilon)$ and a continuously differentiable, real valued function $\phi$ defined on this set such that $\phi(0) = 0$ and $f(x, \phi(x)) = 0$.

(b) Show that the vector $\langle 1, \phi'(x) \rangle$ is orthogonal to the vector $\langle f_x(x, \phi(x)), f_y(x, \phi(x)) \rangle$ for all $x \in (-\epsilon, \epsilon)$.

(c) Now define the map $F(x, w) = (x + w f_x(x, \phi(x)), \phi(x) + w f_y(x, \phi(x)))$. Show that $F$ is one-to-one in a neighborhood of the origin.