Department of Mathematics and Statistics
University of New Mexico

Real Analysis **Qualifying Exam**

August 2009

Instructions: Complete seven out of the eight problems in the exam to get full credit. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let $f \in \mathcal{C}([0,1])$. Show that $\displaystyle\lim_{n\to\infty} \frac{\int_0^1 x^n f(x)dx}{\int_0^1 x^n dx} = f(1)$.

2. a) Show that the function $f(x) = \sin\left(\frac{\pi}{x}\right)$ is continuous on the interval $(0,1)$.

b) Is f uniformly continuous on $(0,1)$?

c) For a real valued function g defined on a metric apace (X,d) let

$$\omega(r) = \sup\{|g(x) - g(x')| \, : \, d(x,x') \leq r\}.$$

Show that g is a uniformly continuous function iff $\lim_{r\to 0} \omega(r) = 0$.

3. Show that any open cover of the interval $[0,1]$ by open intervals in $[0,1]$ contains a subcover of total length less than or equal to 2. The total length of the subcover is the sum of the lengths of the intervals in the subcover.

4. a) The projection map $p : \mathbb{R}^2 \to \mathbb{R}$ is given by $p(x,y) = x$. Determine if the projection map is (i) a continuous map, (ii) an open map, or (iii) a closed map. We are using the standard topologies in the corresponding Euclidean spaces.

(b) Let $f : X \to Y$ be a continuous map between the metric spaces (X,d) and (Y,ρ). If $K \subset X$ is compact, is it true that $f(K)$ is compact? Explain your answers.

5. a) Let A be an $n \times n$ real valued matrix, and $x \in \mathbb{R}^n$, show that

$$\|Ax\| \leq \|A\| \, \|x\|.$$

Where $||.||$ denotes the Euclidean norm in the corresponding Euclidean spaces, more precisely, if $A = [a_{ij}]_{i,j=1}^n$, and $x = [x_1, \ldots, x_n]^t$, then $\|A\|^2 = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2, \qquad \|x\|^2 = \sum_{j=1}^n |x_j|^2.$

b) Let E be an open subset of $\mathbb{R}^n$ such that every two points $x,\, y \in E$ can be joined by a smooth curve of finite length less than or equal to $100\,||x-y||$. Show that if f is a continuously differentiable map $f : E \to \mathbb{R}^n$, such that, for some constant M we have $||f'(x)|| \leq M$ for all $x \in E$, then f is uniformly Lipschitz, i.e., for some constant Λ we have $||f(x) - f(y)|| \leq \Lambda ||x-y||$, where $||.||$ denotes the Euclidean norm in the corresponding Euclidean spaces.

6. Let E be an open subset of $\mathbb{R}^N$. Assume that the following is a metric on the space of real-valued continuously differentiable functions defined on E, $\mathcal{C}^1(E:\mathbb{R})$,

$$d(f,g) \equiv \sum_{n=1}^{\infty} 2^{-n} \frac{||f-g||_{K_n}}{1+||f-g||_{K_n}},$$

where K_n is a sequence of compact sets with $K_n \subset\subset K_{n+1}$ (i.e. K_n is contained in the interior of K_{n+1}), $\cup_{n=1}^{\infty} K_n = E$ and for a function $f \in \mathcal{C}^1(E:\mathbb{R})$ and a compact $K \subset E$ we let

$$||f||_K = \sup_{x\in K}\left(|f(x)| + \sum_{j=1}^{N} |D_j f(x)|\right).$$

a) Show that the above metric defines a topology in which convergence means uniform convergence over any compact subset of a function and its first derivatives.

b) Show that $\mathcal{C}^1(E:\mathbb{R})$ is complete with respect to the defined metric.

7. Let $\psi \in \mathcal{C}^1(\mathbb{R}^2:\mathbb{R})$ be a function with nowhere vanishing gradient, $\psi = \psi(u,v)$ and $a,b \in \mathbb{R}$ two constants, such that, $a\frac{\partial\psi}{\partial u} + b\frac{\partial\psi}{\partial v} \neq 0$.

a) Show that the equation $\psi(x+az, y+bz) = 0$ defines z implicitly as a function of (x,y), $z = z(x,y) \in \mathbb{R}$.

b) Show that $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = -1$.

8. Let $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ are 3-D vector fields depending on the time variable t and

$$\omega = E_1 dx\wedge dt + E_2 dy \wedge dt + E_3 dz\wedge dt + B_1 dy\wedge dz + B_2 dz\wedge dx + B_3 dx\wedge dy.$$

Show that if ω is a closed differential two form in $\mathbb{R}^4$, then:

a) $\dfrac{d\mathbf{B}}{dt} + \operatorname{curl}\mathbf{E} = 0$ and $\operatorname{div}\mathbf{B} = 0$ (curl and div are taken in the space variables (x,y,z) only);

b) at any fixed moment t, the field $\mathbf{B}$ has zero flux through any closed smooth surface Σ in $\mathbb{R}^3$ - the (x,y,z) space;

c) show the following relation, valid at any fixed moment t, between the circulation of E along a closed curve γ and the flux of $\mathbf{B}$ through the smooth surface Σ with boundary γ,

$$\int_\gamma E_1\,dx + E_2\,dy + E_3\,dz \;=\; -\frac{d}{dt}\int_\Sigma B_1 dy\wedge dz + B_2 dz\wedge dx + B_3 dx\wedge dy.$$

(Both, the curve and the surface are in in $\mathbb{R}^3$ - the (x,y,z) space.)