Real Analysis Qualifying Exam
August 12, 2002

Instructions: *There are eight problems, please do all of them. Start each problem on a new sheet of paper and write on one side of each sheet of paper. Remember to write the last four digits of your Social Security number in all pages and to clearly number them. Good luck!!*

Problem 1: Given $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$, define $\|x\|_p = (\sum_{i=1}^{d} |x_i|^p)^{1/p}$, and $\|x\|_\infty = \sup_{i=1,\ldots,d} |x_i|$.

Show that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

Problem 2: A real-valued function on an open interval in $\mathbf{R}$ is called *convex* if no point on the line segment between any two points of its graph lies below the graph. Show that if the function is differentiable and convex, then no point of the graph lies below any point of any tangent line to the graph. State these conditions in precise analytical terms and prove them. (You might try a direct proof or a proof by contradiction!)

Problem 3: Consider the space X of sequences $s = (x_n)_{n\in\mathbf{N}}$ in $\mathbf{R}$ such that $\sup_{n\in\mathbf{N}} |x_n| < \infty$ (bounded sequences or ℓ^∞). Introduce the distance function between two sequences $s^i = (x_n^i)_{n\in\mathbf{N}}$, $i = 1, 2$,

$$d(s^1, s^2) = \sup_{n\in\mathbf{N}} |x_n^1 - x_n^2|.$$

(a) Show that (X, d) is a metric space and is complete.

(b) Consider the subspace Y of X of sequences converging to zero. Clearly (Y, d) is a metric space, show that it is complete but not compact.

Problem 4: The space $C([a, b])$ of real-valued continuous functions over the interval $[a, b]$ is a complete metric space with the distance induced by the sup norm,

$$d(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

(a) Given a real-valued function F defined on $C([a, b])$, define continuity and uniform continuity of F.

(b) Let $F : C([a, b]) \to \mathbf{R}$ be given by $F(f) = \int_a^b f(t)dt$. Show that F is uniformly continuous.

Problem 5: Let $\{f_n\}_{n\in\mathbf{N}}$ be a uniformly convergent sequence of continuous real-valued functions on the compact interval $[a, b]$. Show that

$$\lim_{n\to\infty} \left(\int_a^b f_n(x)\, dx \right) = \int_a^b \lim_{n\to\infty} f_n(x)\, dx.$$

You first have to argue why the limit function is integrable!

Problem 6: Suppose that the real-valued function $g(x, y)$ is such that the mixed second order partial derivatives, $\frac{\partial^2 g}{\partial x \partial y}$ and $\frac{\partial^2 g}{\partial y \partial x}$, are continuous on an open subset U of $\mathbf{R}^2$. Show that

$$\frac{\partial^2 g}{\partial x \partial y}(x, y) = \frac{\partial^2 g}{\partial y \partial x}(x, y), \quad \forall (x, y) \in U.$$

Hint: Show that, for all rectangles $[a, b] \times [c, d] \subset U$,

$$\int_a^b \int_c^d \frac{\partial^2 g}{\partial x \partial y}(x, y)dy\, dx = \int_a^b \int_c^d \frac{\partial^2 g}{\partial y \partial x}(x, y)dy\, dx.$$

In problems 7 and 8 consider the following set up.

Let $M : \mathbf{R}^2 \to \mathbf{R}^2$, $M(x, y) = (x', y')$, be given by,

$$x' = x + P(x, y), \qquad P(x, y) = (x + y)^2; \tag{1}$$
$$y' = y + Q(x, y), \qquad Q(x, y) = -(x + y)^2. \tag{2}$$

Let $I(r) = (-r, r)$, $\overline{I}(r) = [-r, r]$ and $S(r) = I(r) \times I(r)$.

Problem 7: State the contraction mapping principle and use it to show that for each $(x', y) \in S(r)$ there is a unique $x \in \overline{I}(2r)$ satisfying (1) for sufficiently small r. That is, show that there exists $r_0 > 0$, such that for all $0 < r \leq r_0$, and for each $(x', y) \in S(r)$ there exists a unique $x \in \overline{I}(2r)$ such that

$$x = T_{x',y}(x), \quad \text{where} \quad T_{x',y}(x) = x' - P(x, y).$$

Remark: (1) can be solved directly for x, however the point of the problem is to show you know how to apply the contraction mapping principle.

Problem 8: For r small enough, there is a unique solution $x \in \overline{I}(2r)$ of (1), of the form $x = x' + F(x', y)$, for $(x', y) \in S(r)$, and with $F \in C^1(S(r))$. Thus the map (1)-(2) can be written implicitly as

$$\begin{aligned} x &= x' + F(x', y) \\ y' &= y + H(x', y), \qquad H(x', y) = Q(x' + F(x', y), y). \end{aligned}$$

State and use Stokes' theorem in $\mathbf{R}^2$ to show that there exists a function $G : S(r) \to \mathbf{R}$ such that

$$D_1 G(x', y) = H(x', y), \quad \text{and} \quad D_2 G(x', y) = F(x', y). \tag{3}$$

That is, argue that $\int_C H(x', y)dx' + F(x', y)dy$ is independent of the path C connecting $(0, 0)$ and (x'_0, y_0), and that $G(x'_0, y_0) = \int_{(0,0)}^{(x'_0, y_0)} H dx' + F dy$ makes sense as a definition of the function G satisfying (3).