Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam**

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Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let $f \in \mathcal{C}([0,1])$. Show that $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$.

2. Let $|a| < 1$ and $f_n(t) = \sin\left((2n+1)\frac{\pi}{2}t\right)$, $t \in \mathbb{R}$.

(a) Show that for any a and t as above, the series $\sum_{n=1}^{\infty} a^{2n+1} f_n(t)$ converges absolutely.

(b) Determine if $f(t) = \sum_{n=1}^{\infty} a^{2n+1} f_n(t)$ is a continuous function of t.

(c) Show that $\frac{1}{\pi} \ln \frac{1+a}{1-a} - \frac{2}{\pi} a = \int_0^1 f(t)\, dt$.

3. (Lebesgue covering theorem) Let (K,d) be a compact metric space and $\mathfrak{U}$ an open cover of K. Show that there is an $\epsilon > 0$ such that $\{B(x,\epsilon)\}_{x\in K}$ is a refinement of $\mathfrak{U}$, i.e., every $B(x,\epsilon)$ is contained in some open set from $\mathfrak{U}$.

4. Let (X,d) be a compact metric space and F a mapping $F : X \to X$ such that

$$d(F(x),F(y)) < d(x,y) \quad \text{for all} \quad x,y \in X.$$

(a) Show that the function $g : X \to [0,\infty)$, defined by $g(x) = d(x,F(x))$, is a continuous function, whose minimum must be zero.

(b) Show that the equation $F(x) = x$ has exactly one solution, i.e., F has exactly one fixed point.

5. (a) State the inverse function theorem on $\mathbb{R}^n$.

(b) Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable map such that its Jacobian is non-zero at every point and F is proper (i.e. the pre-image of any compact set is a compact set). Show that F is onto. Consider the exponential function $f(x) = e^x$, $f : \mathbb{R} \to \mathbb{R}$, is there a contradiction with the previous statement? Explain.

6. An electric charge q located at the origin produces the electric field

$$\vec{E} = \frac{q\vec{R}}{4\pi\epsilon\|\vec{R}\|^3},$$

where $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$ and ϵ is a physical constant, called the electric permittivity.

(a) Show that

$$\int\int_S \vec{E} \cdot \vec{N}\, dS = 0$$

if the closed surface S does not enclose the origin (we are assuming S to be a piecewise smooth surface bounding a bounded domain in $\mathbb{R}^3$, and $\vec{N}$ is the outer normal to the surface S).

(b) Show that

$$\int\int_S \vec{E} \cdot \vec{N}\, dS = \frac{q}{\epsilon}$$

if the closed surface S encloses the origin.

7. Let $\vec{F} = y\vec{i} - x\vec{j} + z\vec{k}$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$

(a) along the segment C joining $(1, 0, 0)$ to $(1, 0, 4)$;

(b) along the helix C given by $x = \cos t$, $y = \sin t$, $z = \frac{4t}{2\pi}$, for $0 \leq t \leq 2\pi$.

(c) Decide whether $\vec{F}$ is a conservative vector field or not.