Statistics Masters and Ph.D. Qualifying Exam
In Class: Tuesday August 16, 2005

Instructions: The exam has 5 multi-part problems of equal value. All of the problems will be graded. Write your ID number on your answer sheets (last 4 digits of your SSN). Do not put your name on any of the sheets. Be clear, concise, and complete.

1. Let \( X \) be the number obtained from a single roll of a fair six-sided die. Given the value of \( X = x \), roll a second fair die with \( x \) sides, numbered 1, 2, ..., \( x \). Let \( Y \) denote the number obtained on the roll of the second die.

(a) Find the joint probability function for \( X \) and \( Y \).
(b) Are the random variables \( X \) and \( Y \) independent? Justify your answer.
(c) Evaluate \( P(Y > X - 2) \).
(d) Compute \( E[Y|X = x] \) and \( Var[Y|X = x] \).
(e) Use (d) to evaluate \( E[Y] \).

2. Consider a sequence of days, and let \( R_i \) denote the event that it rains on day \( i \), \( i = 0, 1, ... \). Similarly, let \( N_i \) denote the event "no rain" on day \( i \). Assume that \( P(R_i|R_{i-1}) = \alpha \) and \( P(N_i|N_{i-1}) = \beta \) for all \( i > 0 \) and that day 0 is today. Suppose further that only day \( i - 1 \)'s weather is relevant to predicting day \( i \)'s; that is, expressions like \( P(R_i|R_{i-1} \cap R_{i-2} \cap ... \cap R_0) \) are equivalent to \( P(R_i|R_{i-1}) \).

(a) If the probability of rain today (day 0) is \( p \), what is the probability of rain tomorrow?
(b) What is the probability of rain the day after tomorrow?
(c) What is the probability of rain \( n \) days from now?
(d) (Extra Credit) What happens in (c) as \( n \) approaches infinity?

3. Let \( X_1 \) and \( X_2 \) be independent and identically distributed random variables with a \( U(0,1) \) distribution.

(a) Find the cumulative distribution function (cdf) of \( Z = X_1 X_2 \). Hint: You might first consider finding the distribution function of \( \log(Z) \).
(b) Use part (a) to find the density function of \( Z \).
(c) Let \( Y = X_1 + X_2 \). Find the conditional probability that \( Y \leq \frac{1}{2} \), given that \( Z \leq \frac{1}{16} \).

4. Let \( X_1, \ldots, X_n \sim f_X(x|\theta) = \frac{x^2}{\theta^3} \sqrt{\frac{2}{\pi}} \exp \left( -\frac{x^2}{2\theta^2} \right) \) for \( x \geq 0 \) and \( \theta > 0 \). This is called the Maxwell distribution. Note that \( E(X_1) = \theta \sqrt{\frac{8}{\pi}} \) and \( E(X_i^2) = 3\theta^2 \).

(a) Find a complete, sufficient statistic for \( \theta \).
(b) What is the expectation of the statistic found in part (a)?
(c) Find an uniformly minimum variance unbiased estimator (UMVUE) for \( \theta^0 \).
(d) Find the maximum likelihood estimator (MLE) for $\theta$; call it $\hat{\theta}$. Find the MLE for $\theta^2$.
(e) Find the Cramer-Rao lower bound (CRLB) for unbiased estimators of $\theta^2$.
(f) Does the estimator in part (c) achieve the CRLB for estimating $\theta^2$?

5. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, 1)$.

(a) Find a uniformly most powerful (UMP) test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for $\theta_0 < \theta_1$. State the rejection region $R$ in terms of the significance level $\alpha$ and a sufficient statistic for $\theta$.

(b) Find the likelihood ratio test (LRT) statistic $\lambda(x)$ for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

(c) What is the exact distribution of $-2 \log \lambda(X)$ when $H_0$ is true?

(d) Say that $-2 \log \lambda(x) = 14.3$ is observed. Do you reject or not reject $H_0$ at the $\alpha = 0.05$ level? Note: $P(X^2_1 < 3.84) = 0.95$. 

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