Statistics Qualifying Exam: In Class Portion
Due on 2012-01-12

Please complete the problems on a separate sheet of paper with your name at the top. Make sure to show your work and/or provide an explanation for each problem. Partial credit will be given when merited. Note: This assignment may seem long, but we will have only four homework sets this quarter.

Problem 1
Assume that we have observed a random sample \(x_1, x_2, \ldots, x_n\) from a \(N(\mu, 1)\) population. We are interested in testing the Hypotheses \(H_0: \mu = 0\) vs \(H_1: \mu \neq 0\). Assume that \(n = 30\) and we have observed \(\bar{x} = 1.35\).

(a) What is the form of the UMPU test with a fixed \(\alpha = .05\)?
(b) What is the p-value for this test?
(c) To conduct a Bayesian version of this test assume the following mixture prior for \(\mu\).

\[
\pi(\mu) = \begin{cases} 
0.5 & \text{if } \mu = 0 \\
0.5 \star \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{1}{2}\mu^2\right] & \text{if } \mu \neq 0
\end{cases}
\]

What is the posterior probability that \(\mu\) is equal to zero?
(d) Compare and contrast your answers in parts (b) and (c).

Problem 2
Assume that a random variable \(X\) has the following cumulative density function (cdf) given \(\theta\):

\[
F_X(x|\theta) = \begin{cases} 
0 & \text{if } x < 1 \\
\frac{\log(x)}{\log(\theta)} & \text{if } 1 \leq x < \theta \\
1 & \text{if } \theta \leq x
\end{cases}
\]

where \(\theta \in \{2, 3, \ldots\}\) and \(\log(x)\) denotes the natural logarithm of \(x\). Define the following two random variables:

\[Y = \lfloor X \rfloor = \text{the integer part of } X\]
\[W = \langle X \rangle = \text{the fractional part of } X.\]

For example, \(\lfloor 1.5 \rfloor = 1, \langle 1.5 \rangle = .5\).

(a) Find the CDF of \(W, F_W(w)\).
(b) Find the PDF of $W$, $f_W(w)$.
(c) Find the mean of $W$.
(d) Find the joint distribution of $Y$ and $W$.

**Problem 3**

Let the random variable $X$ have the following CDF:

$$F_X(x|\alpha, \beta) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{x^\beta}{\alpha^\beta + x^\beta} & \text{if } x \geq 0.
\end{cases}$$

Where $\alpha > 0$ and $\beta > 1$.

(a) Verify that $F$ is a valid CDF.
(b) Find the density, $f_X(x|\alpha, \beta)$, of $X$.
(c) Find the median of $X$?
(d) Find $E[X^k]$ for $k < \beta$.
(e) Find the distribution of $Y = \log\left(\left(\frac{X}{\alpha}\right)^\beta\right)$.

(f) Let $X_1, X_2, \ldots, X_n$ denote a random sample of size $n$ from $F$. Find the limiting distribution of $n^{1/\beta} X^{(1)}$.

(g) Let $X_1, X_2, \ldots, X_n$ denote a random sample of size $n$ from $F$. Find the limiting distribution of $\bar{X}$ when $\beta > 3$. If you did not answer (b), write the limiting distribution in terms of the mean and variance of $X$. 